Controllability

Fast Robots, ECE4160/5160, MAE 4190/5190

E. Farrell Helbling, 2/25/25

Class Action Items

- Lab 3 is due today/tomorrow, if you need to use a slip week, please send us a private message on Ed. You can do this up until the deadline.
- Lab 4 starts today, at the end of this lab you will have a fully-integrated RC car, and we will start thinking about programming simple control strategies!
  - Good example from last year: https://nila-n.github.io/Lab4.html
  - Note about battery connector.
- Grades for Lab 1 and Lab 2 were posted yesterday/ later today, let us know if you have any questions.
  - One thing I will note is that your website serves as a public repository of information, you should write enough text so that we can understand what you worked on (there were a couple examples of videos with no description).

Linear Systems

- Linear systems review
- Eigenvectors and eigenvalues
- Stability
- Discrete time systems
- Linearizing nonlinear systems
- Controllability
- Observability

$$\dot{x} = Ax + Bu$$

These should look familiar from:

- MATH2940 Linear Algebra
- ECE3250 Signals and Systems
- ECE5210 Theory of Linear Systems
- MAE3260 System Dynamics
- and many others…

Based on "Control Bootcamp", Steve Brunton, UW
https://www.youtube.com/watch?v=Pi7l8mMjYVE

Linear Systems Review

- Linear system: $\dot{x} = Ax$
- Solution: $x(t) = e^{At}x(0)$
- Eigenvectors: $T = \begin{bmatrix} \xi_1 & \xi_2 & \dots & \xi_n \end{bmatrix}$
- Eigenvalues: $D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$
  >>[T,D] = eig(A)
- Linear Transform: $AT = TD$
- Solution: $e^{At} = e^{TDT^{-1}t}$
- Mapping from x to z to x: $x(t) = Te^{Dt}T^{-1}x(0)$
- Stability in continuous time: $\lambda = a + ib$, stable iff $a < 0$
- Discrete time: $x(k+1) = \tilde{A}x(k)$, where $\tilde{A} = e^{A\Delta t}$
- Stability in discrete time: $\tilde{\lambda}^n = R^n e^{in\theta}$, stable iff $R < 1$
- Nonlinear systems: $\dot{x} = f(x)$
- Linearization: $\left.\frac{Df}{Dx}\right|_{\bar{x}}$

Linear Systems

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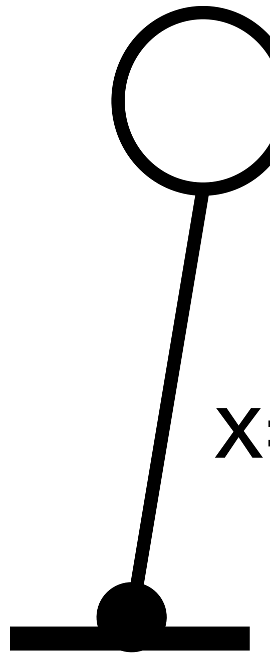
$$\dot{x} = Ax + Bu$$



$$x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$

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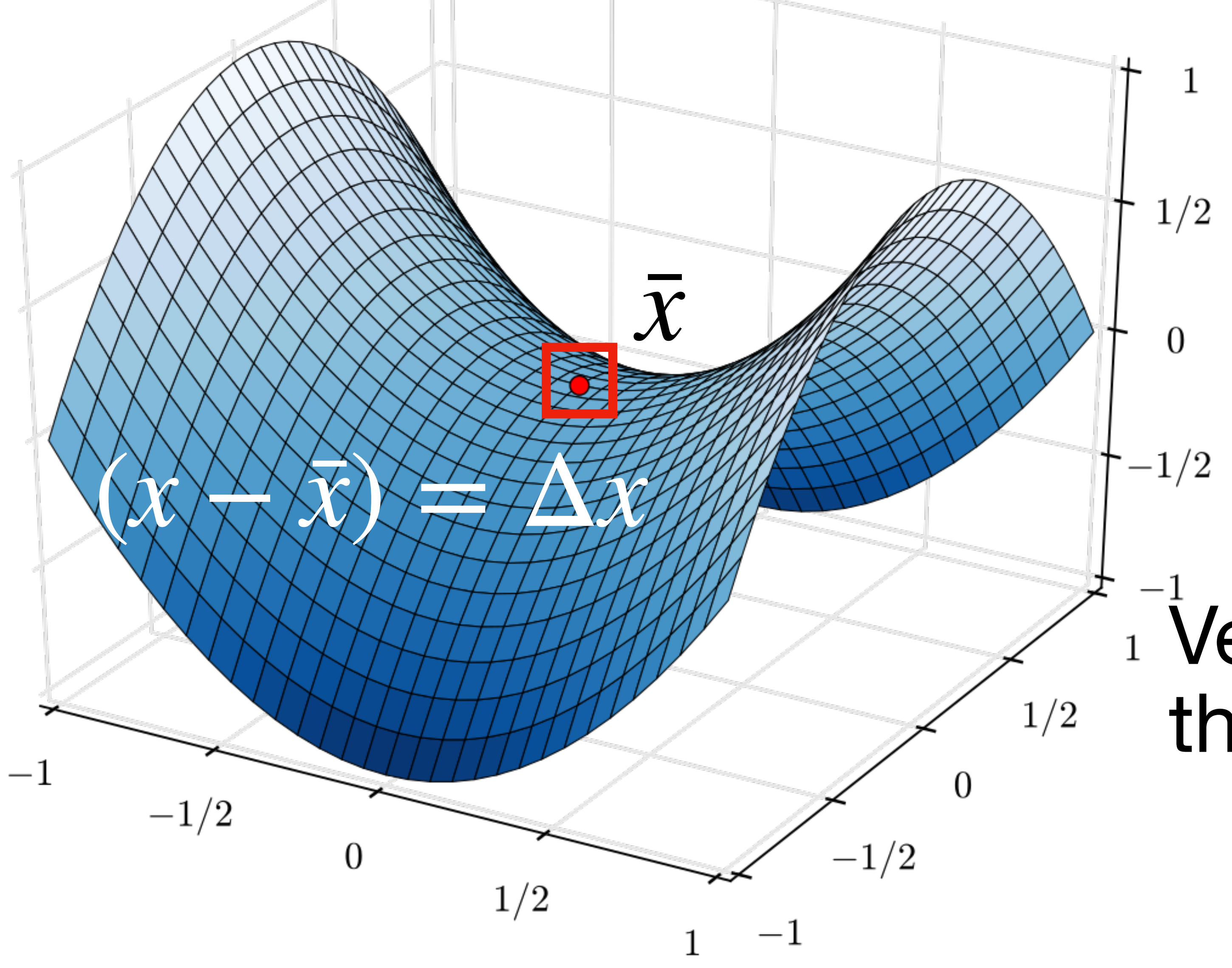
Linearizing Nonlinear Systems

Basic steps to linearize nonlinear systems

$$\dot{x} = f(x) \longrightarrow \dot{x} = Ax$$



Vector field of the dynamics

- Find some fixed points
  - $\bar{x}$ st $f(\bar{x}) = 0$
- Linearize about them
  - $\left.\dfrac{Df}{Dx}\right|_{\bar{x}} = \left[\dfrac{\delta f_i}{\delta x_j}\right]$ "Jacobian"
- If you zoom in on $\bar{x}$, your system will look linear!
- Good control will keep you near the fixed point, where the model is valid!

$$\dot{(x - \bar{x})} = \cancel{f(\bar{x})}^{\,0} + \left.\frac{Df}{Dx}\right|_{\bar{x}}(x-\bar{x}) + \left.\frac{D^2f}{D^2x}\right|_{\bar{x}}(x-\bar{x})^2 + \left.\frac{D^3f}{D^3x}\right|_{\bar{x}}(x-\bar{x})^3 + \ldots$$

<<1

$$\Delta\dot{x} = \left.\frac{Df}{Dx}\right|_{\bar{x}}(\Delta x) \longrightarrow \Delta\dot{x} = A\Delta x$$

Basic steps to linearize nonlinear systems

$$\dot{x} = f(x) \longrightarrow \dot{x} = Ax$$

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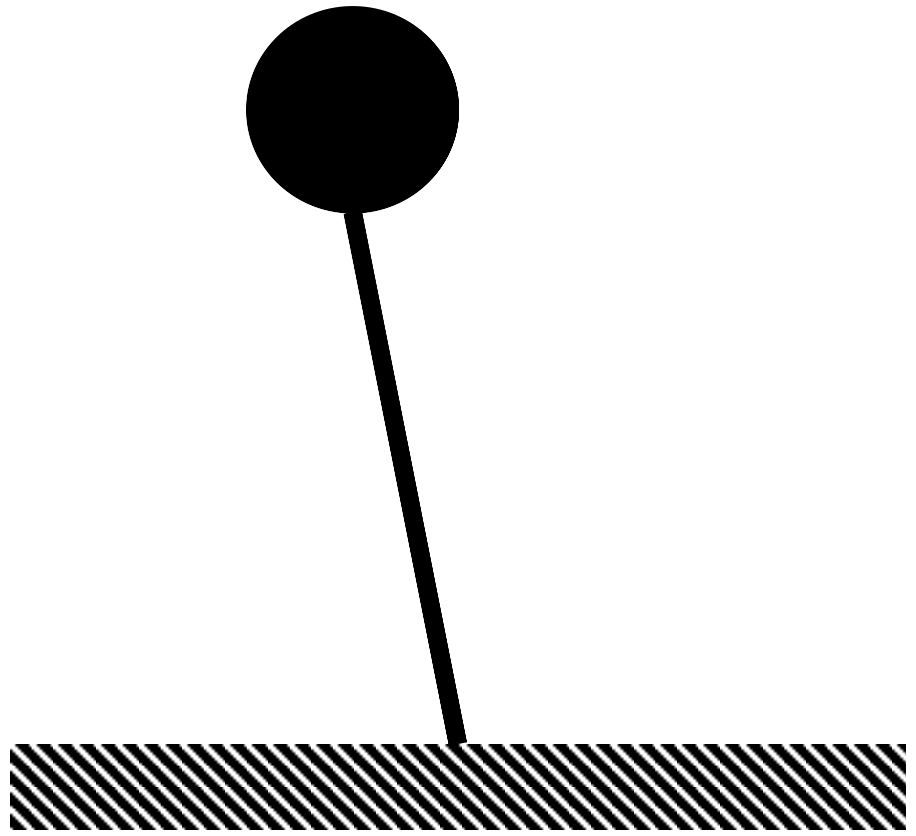
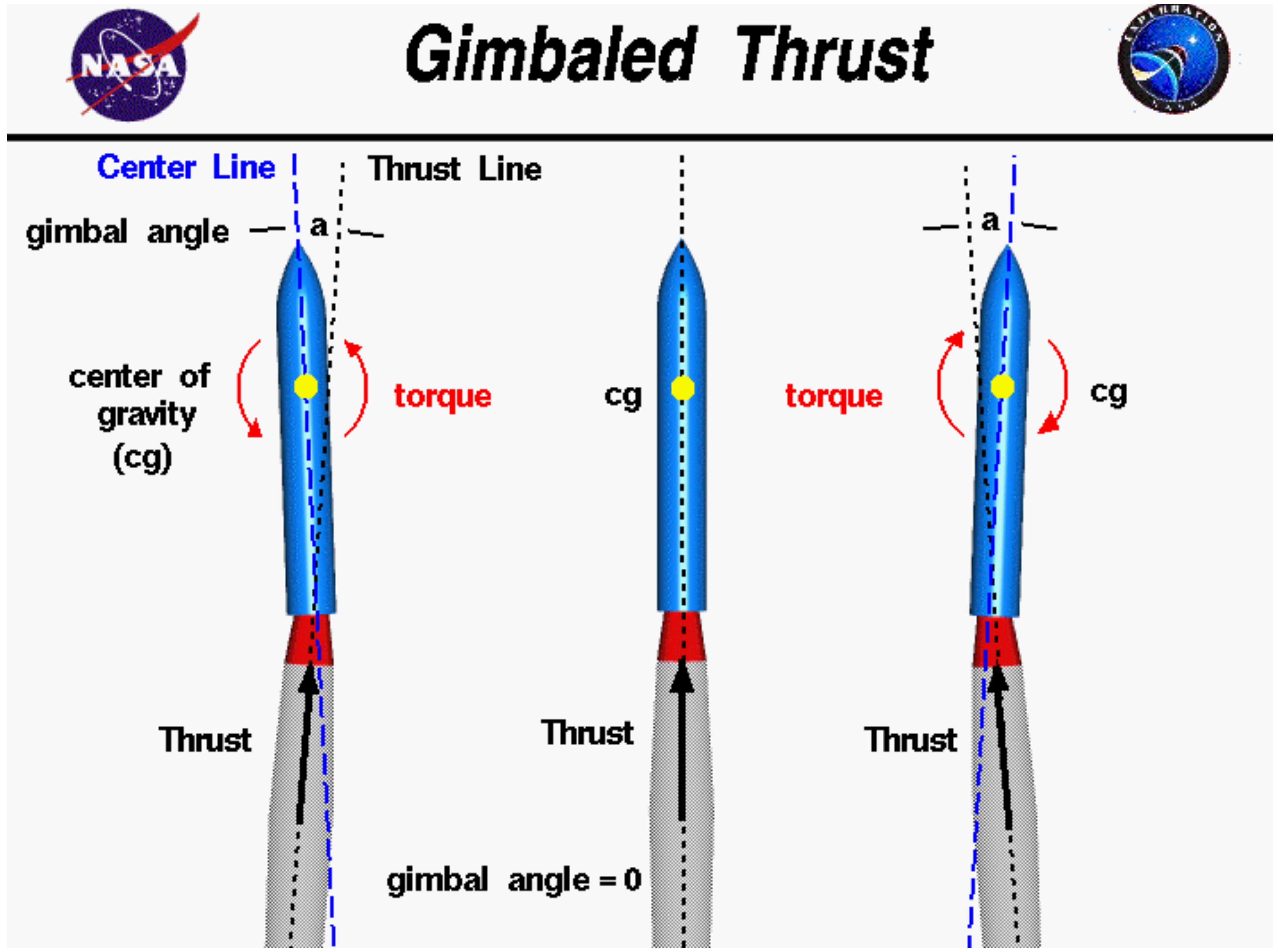


Intuitively, you know:

- Stable point
- Eigenvalues
- Complex poles
- Unstable point



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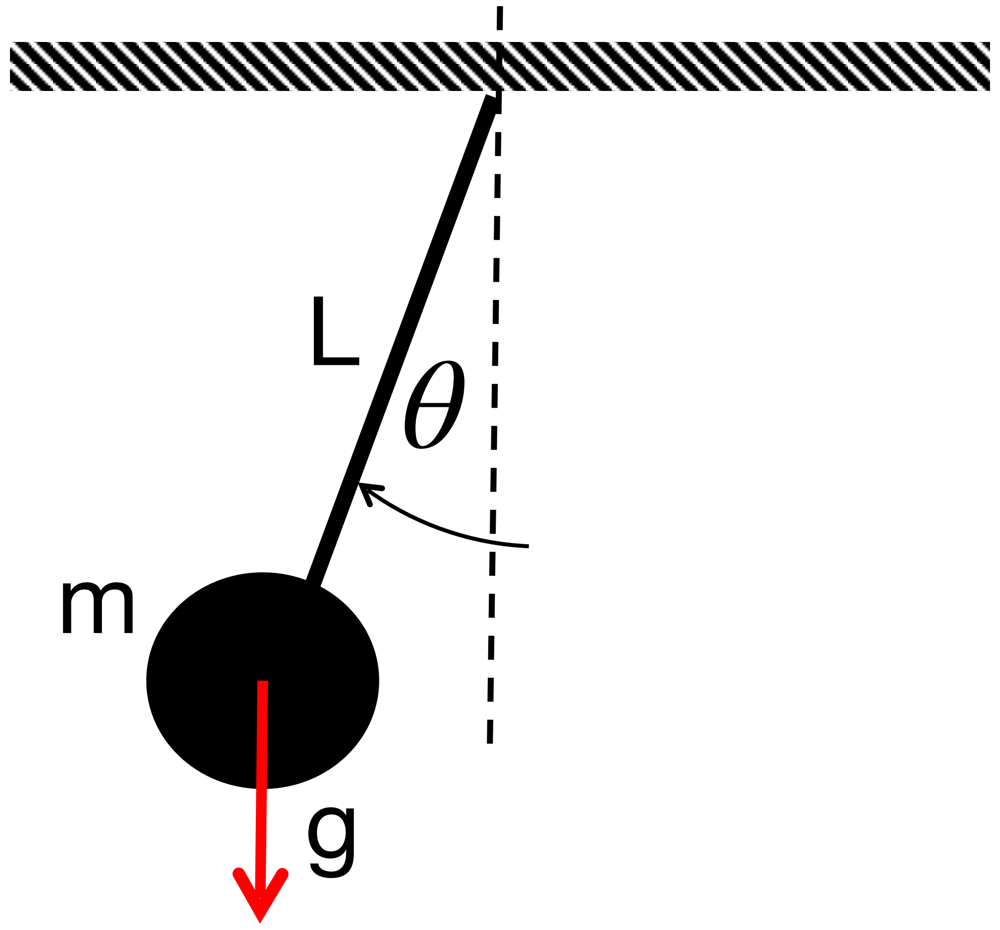

Equations of motion

- $\tau = -mgL\sin(\theta)$
- $\tau = I\ddot{\theta}$
- $I\ddot{\theta} = -mgL\sin(\theta)$
- Point mass inertia: $I = mL^2$
- $mL^2\ddot{\theta} = -mgL\sin(\theta)$
- $\ddot{\theta} = -\frac{g}{L}\sin(\theta)$

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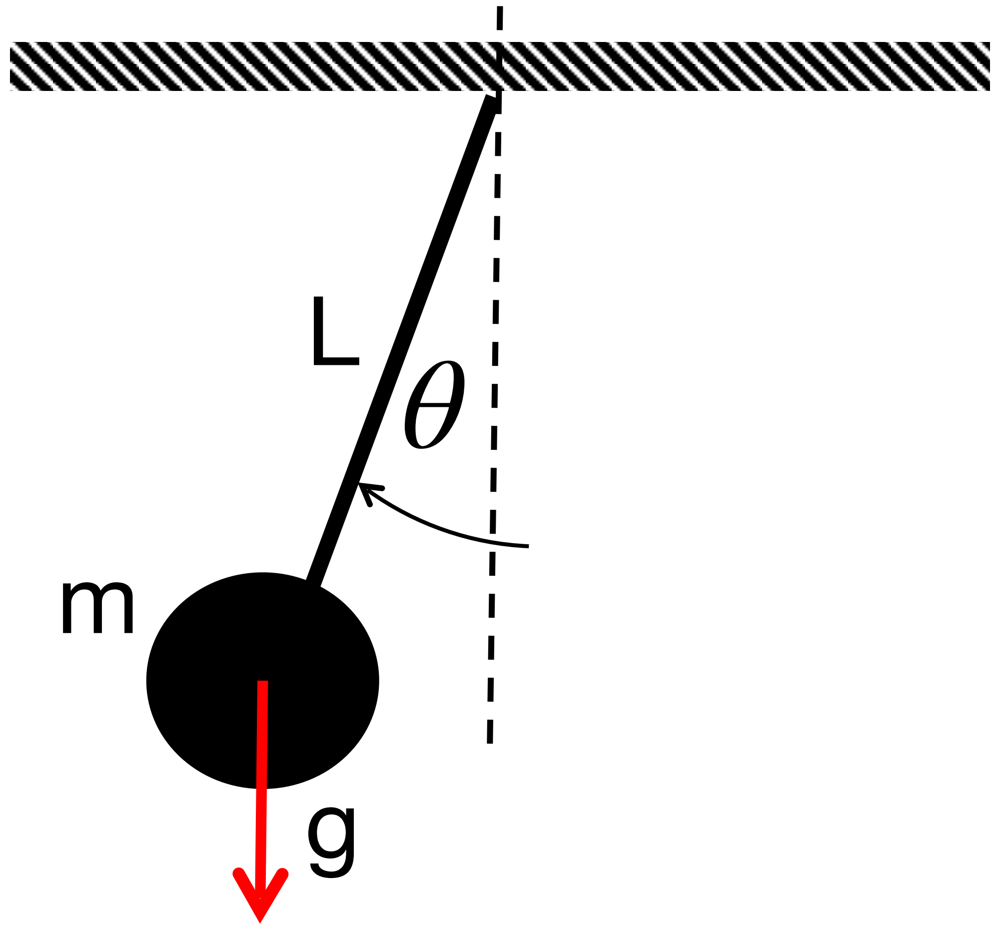

$$\ddot{\theta} = -\frac{g}{L}\sin(\theta) - \delta\dot{\theta}$$

$$\frac{g}{L} = 1$$ **Just simplifies constants**

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$$

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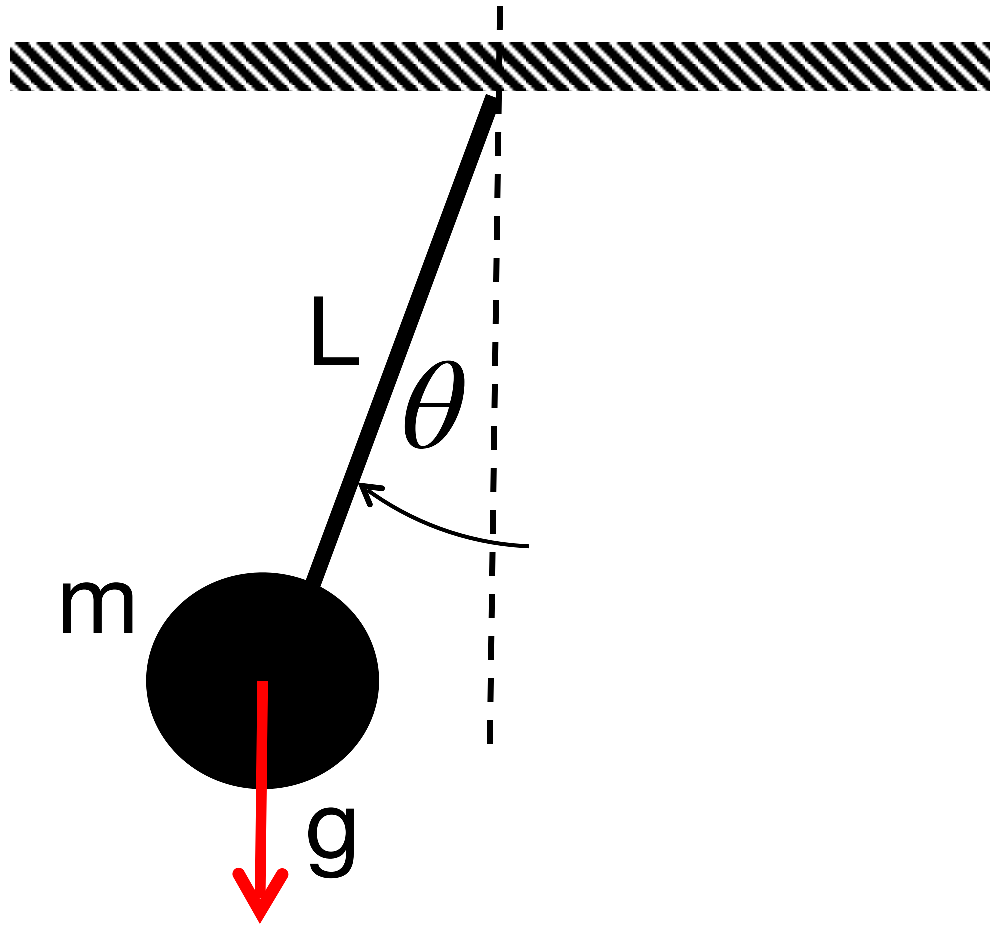
$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\sin(x_1) - \delta x_2 \end{bmatrix}$$

$$\bar{x} = \begin{bmatrix} 0, & \pi \\ & 0 \end{bmatrix} \qquad \frac{Df}{Dx} = \begin{bmatrix} \frac{\delta f_1}{\delta x_1} & \frac{\delta f_1}{\delta x_2} \\ \frac{\delta f_2}{\delta x_1} & \frac{\delta f_2}{\delta x_2} \end{bmatrix}$$



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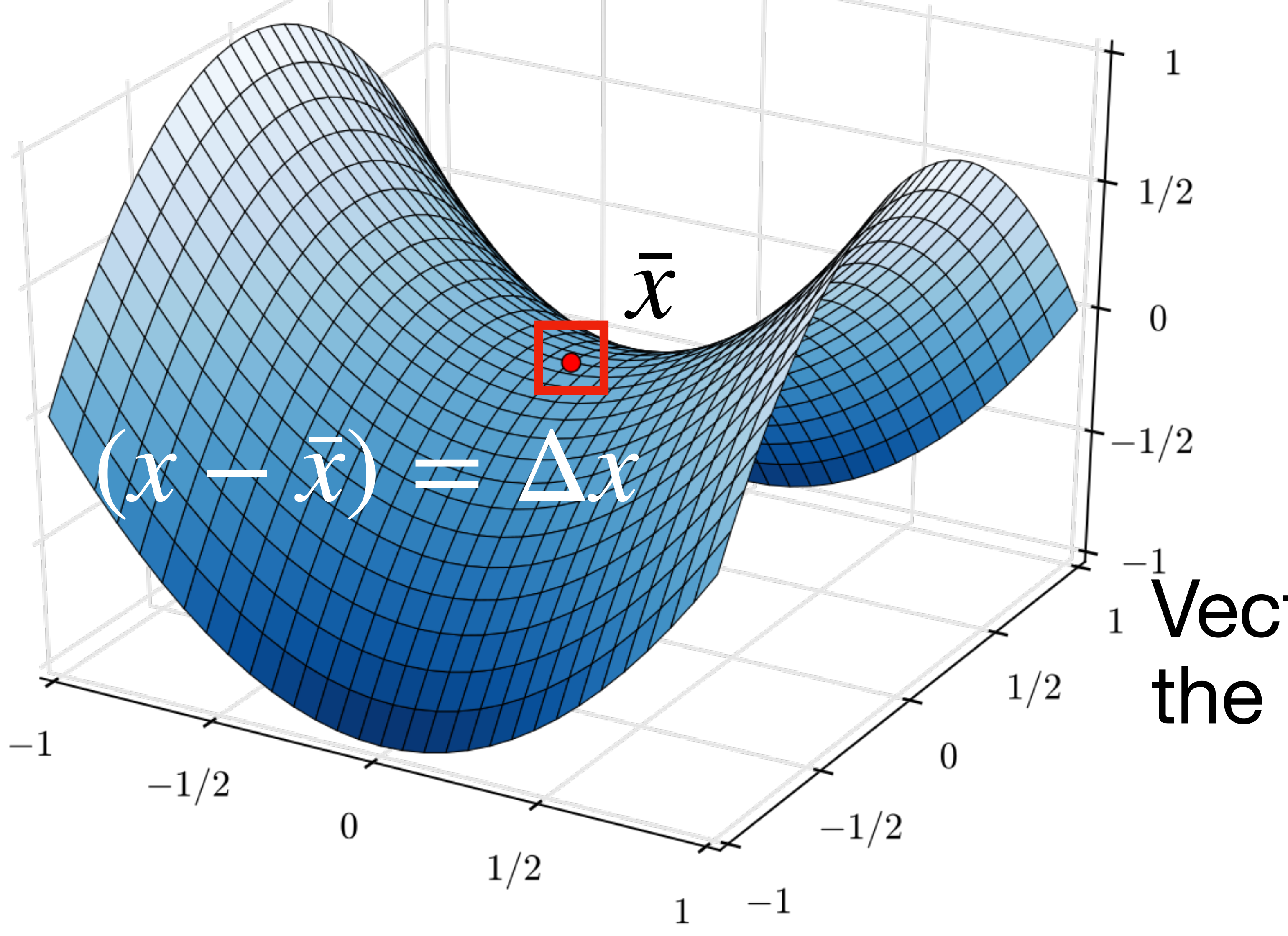

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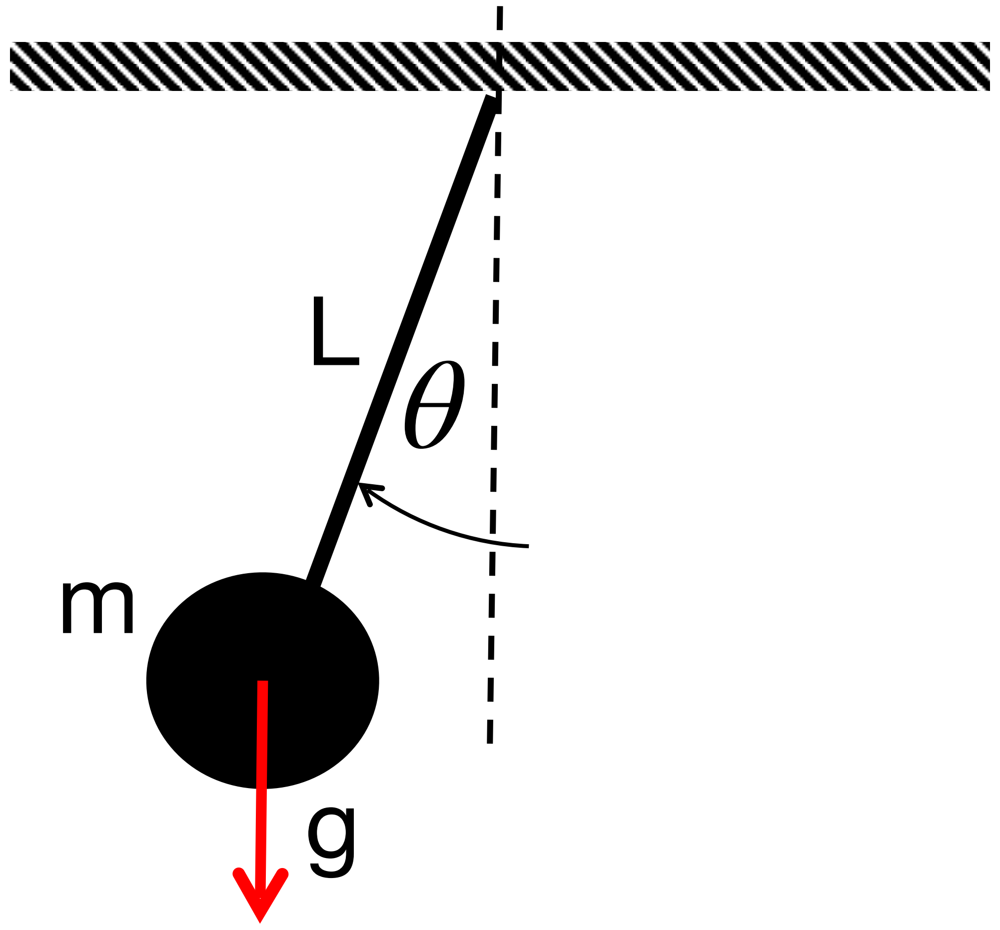


$$A_{down} = \left.\frac{Df}{Dx}\right|_{\bar{x}=[0,0]} = \begin{bmatrix} 0 & 1 \\ -1 & -\delta \end{bmatrix} \qquad \lambda_{down} = -\delta' \pm i$$

stable

$$A_{up} = \left.\frac{Df}{Dx}\right|_{\bar{x}=[\pi,0]} = \begin{bmatrix} 0 & 1 \\ 1 & -\delta \end{bmatrix} \qquad \lambda_{up} = \pm 1$$

unstable

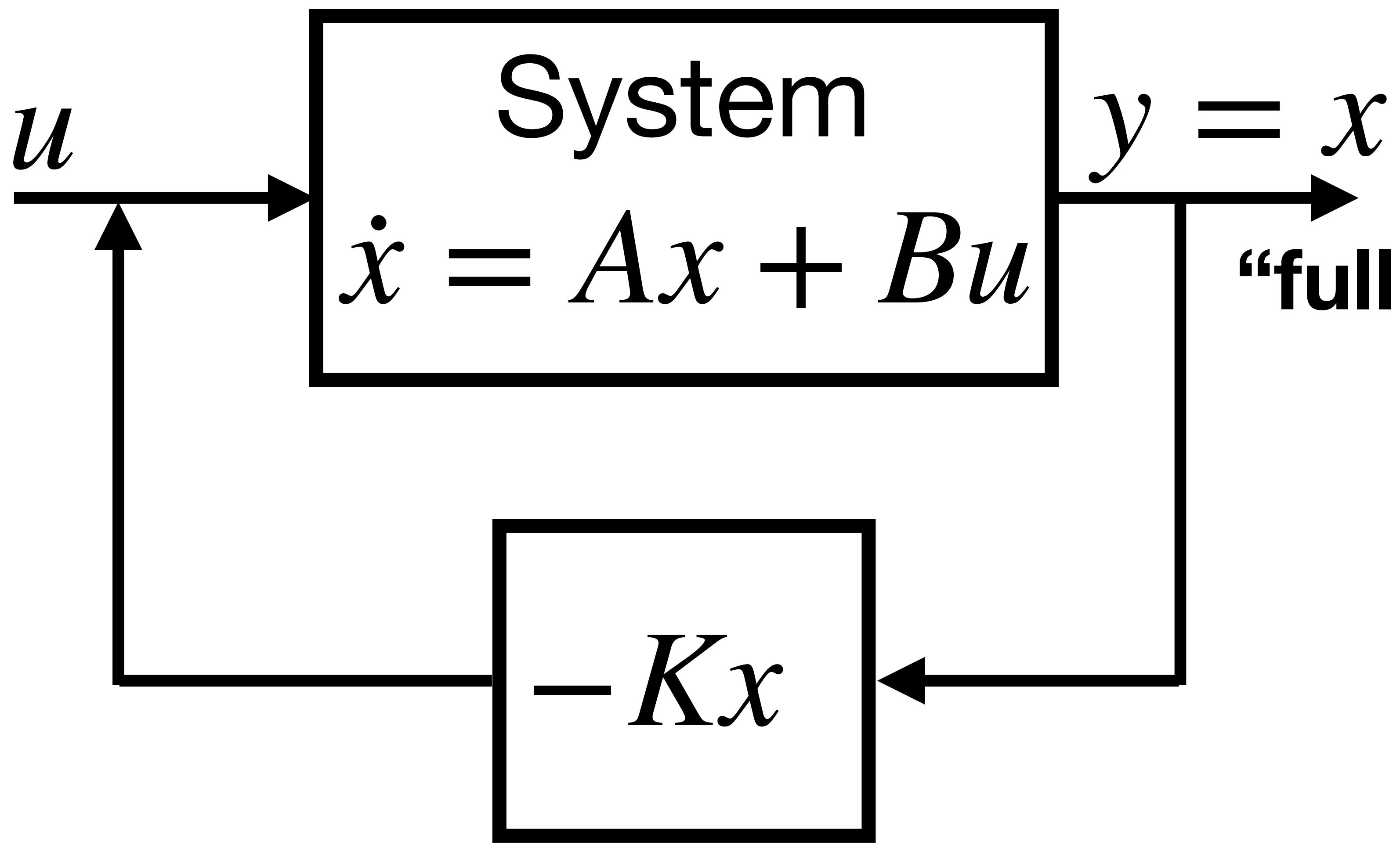
Controllability

Controllability

$$\dot{x} = Ax + Bu$$

$$x \in \mathbb{R}^n$$
$$A \in \mathbb{R}^{n \times m}$$
$$u \in \mathbb{R}^q$$
$$B \in \mathbb{R}^{n \times q}$$

- Is the system controllable?
- How do we design the control law, u?



“full state feedback”

A linear controller (K matrix) can be optimal for linear systems!

Controllability

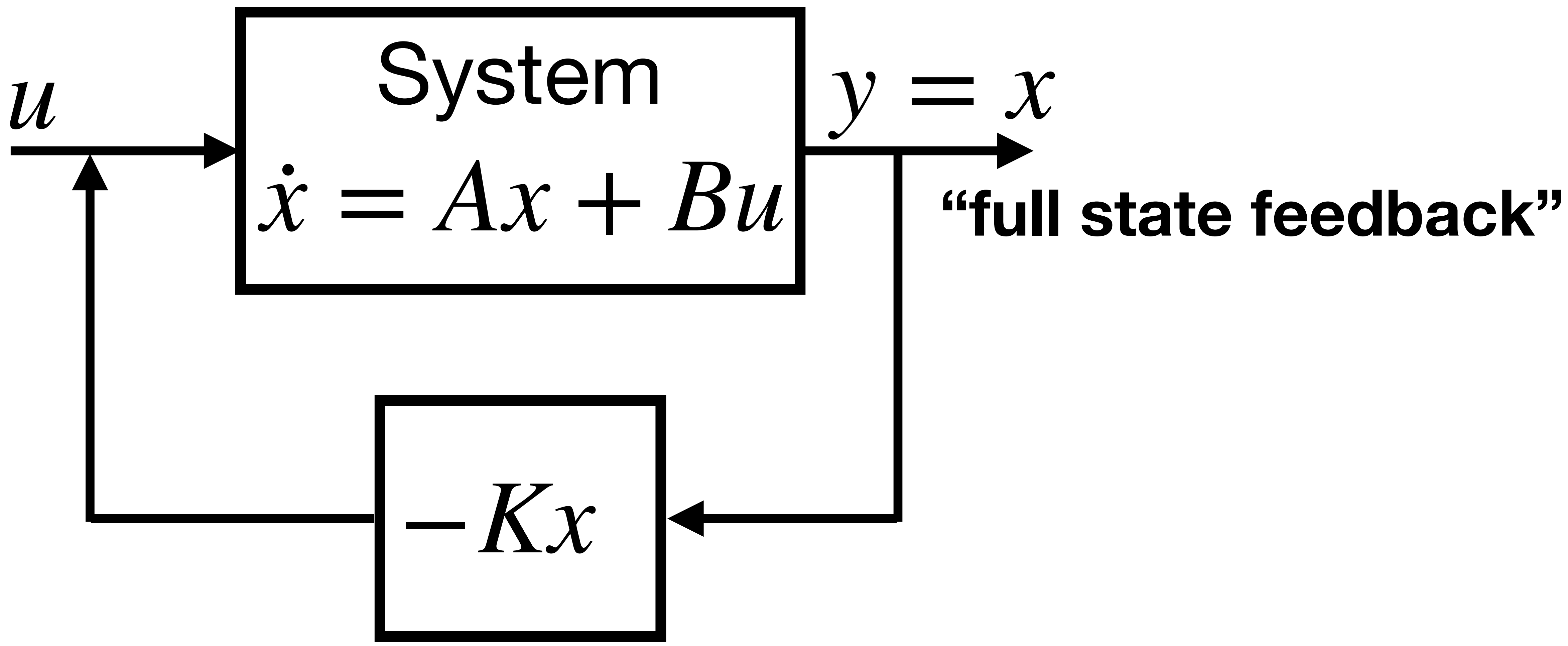
- Is the system controllable?
- How do we design the control law, u?

$$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n$$

$$\dot{x} = Ax - BKx \qquad A \in \mathbb{R}^{n\times m}$$

$$\dot{x} = (A - BK)x \qquad u \in \mathbb{R}^q$$

New dynamics $\qquad B \in \mathbb{R}^{n\times q}$



A linear controller (K matrix) can be optimal for linear systems!

Controllability

- A system is controllable if you can steer your state x anywhere you want in $\mathbb{R}^n$

$$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n$$

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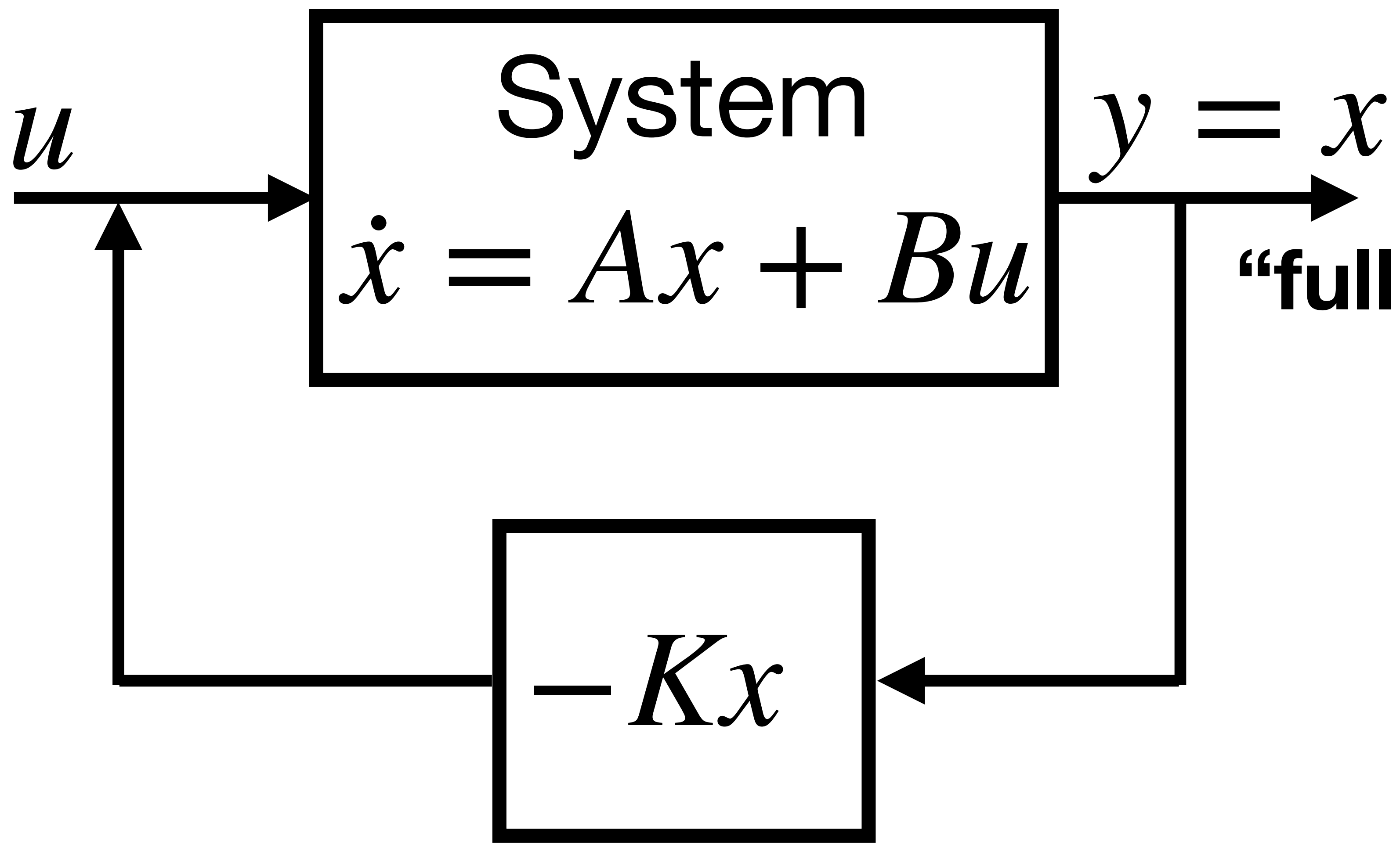

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New dynamics $\qquad B \in \mathbb{R}^{n \times q}$

Often, you don't get to choose A or B

Controllability

$$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n$$

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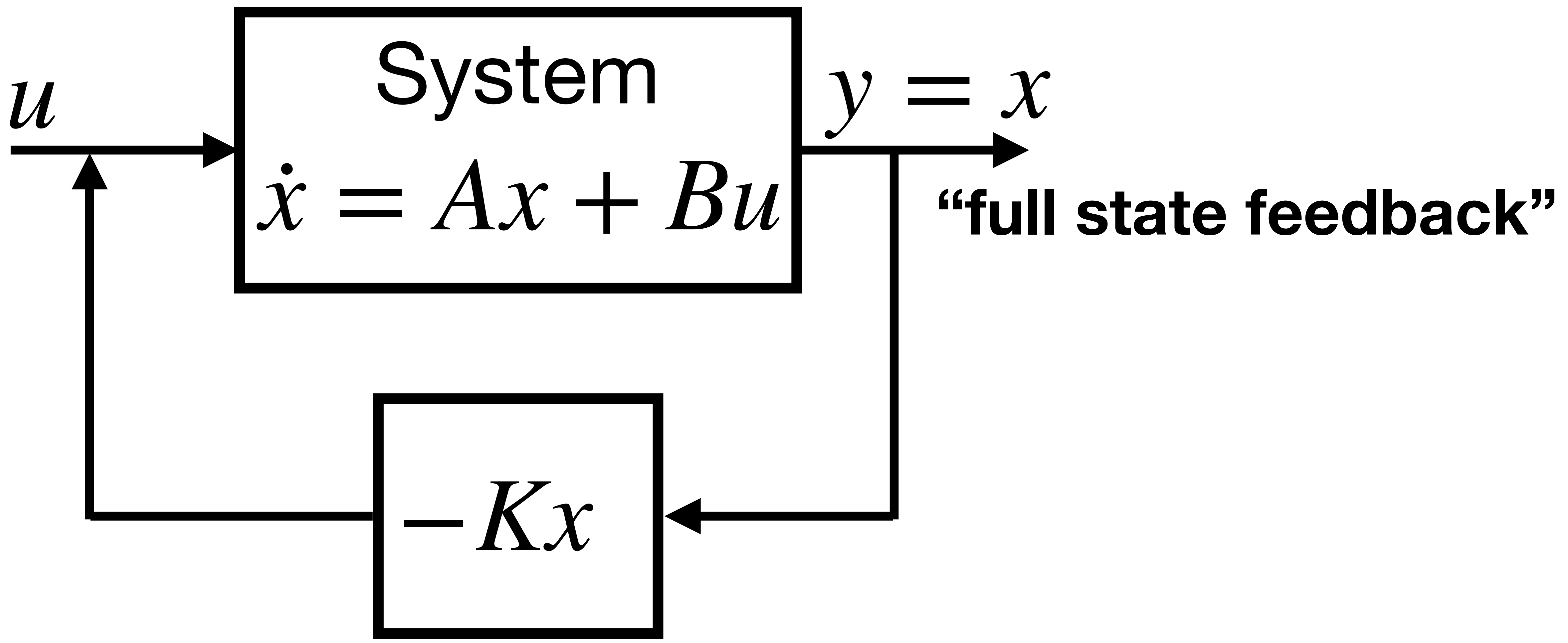
New dynamics $\qquad B \in \mathbb{R}^{n\times q}$

- A system is controllable if you can steer your state x anywhere you want in $\mathbb{R}^n$
- Matlab >> rank(ctrb(A,B))



A linear controller (K matrix) can be optimal for linear systems!

Controllability

- Can you control this system?
  - $$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n$$

$$\dot{x} = Ax - BKx \qquad A \in \mathbb{R}^{n \times m}$$

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- Controllability matrix
  - Matlab >>ctrb(A,B)
  - $\mathbb{C} = \begin{bmatrix}B & AB & A^2B & \dots & A^{n-1}B\end{bmatrix}$
  - The system is controllable iff $\text{rank}(\mathbb{C}) = n$

$$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n$$

$$\dot{x} = Ax - BKx \qquad A \in \mathbb{R}^{n\times m}$$

$$\dot{x} = (A - BK)x \qquad u \in \mathbb{R}^q$$

New dynamics $\qquad B \in \mathbb{R}^{n\times q}$

FYI! Just because a linearized, nonlinear system is uncontrollable, it can still be nonlinearly controllable!

Controllability in Discrete Time

$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n$

$\mathbb{C} = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$

- Why does $\mathbb{C}$ predict controllability?
- Discrete time impulse response: $x(k+1) = \tilde{A}x(k) + \tilde{B}u(k)$

(assume a single input actuator)

$$
\begin{aligned}
u(0) &= 1 & x(0) &= 0 \\
u(1) &= 0 & x(1) &= \tilde{A}x(0) + \tilde{B}u(0) = \tilde{B} \\
u(2) &= 0 & x(2) &= \tilde{A}x(1) + \tilde{B}u(1) = \tilde{A}\tilde{B} \\
u(3) &= 0 & x(3) &= \tilde{A}^2\tilde{B} \\
&\vdots & &\vdots \\
u(m) &= 0 & x(m) &= \tilde{A}^{m-1}\tilde{B}
\end{aligned}
$$

If the system is controllable, then the impulse response affects every state in $\mathbb{R}^n$

Review

- Linear system: $\dot{x} = Ax$
- Solution: $x(t) = e^{At}x(0)$
- Eigenvectors: $T = \begin{bmatrix} \xi_1 & \xi_2 & \dots & \xi_n \end{bmatrix}$
- Eigenvalues: $D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$ `>>[T,D] = eig(A)`
- Linear Transform: $AT = TD$
- Solution: $e^{At} = e^{TDT^{-1}t}$
- Mapping from x to z to x: $x(t) = Te^{Dt}T^{-1}x(0)$
- Stability in continuous time: $\lambda = a + ib$, stable iff $a < 0$
- Discrete time: $x(k+1) = \tilde{A}x(k)$, where $\tilde{A} = e^{A\Delta t}$
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- Nonlinear systems: $\dot{x} = f(x)$
- Linearization: $\left.\frac{Df}{Dx}\right|_{\bar{x}}$
- Controllability: $\dot{x} = (A - BK)x$ `>>rank(ctrb(A,B))`

Reachability

Controllability and Reachability

$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n$

$\mathbb{C} = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$

Reachability

- $\mathcal{R}_t$: states that are reachable at time t
- $\mathcal{R}_t = \xi \in \mathbb{R}^n$ for which there is an input $u(t)$ that makes $x(t) = \xi$

Equivalences

- The system is controllable
  - iff $\text{rank}(\mathbb{C}) = n$
- You can choose K to arbitrarily place the eigenvalues of your closed loop system
  - $\dot{x} = (A + BK)x$
- You can reach anywhere in $\mathbb{R}^n$ in a finite amount of time and energy
  - $\mathcal{R}_t = \mathbb{R}^n$

Controllability and Reachability

$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n$

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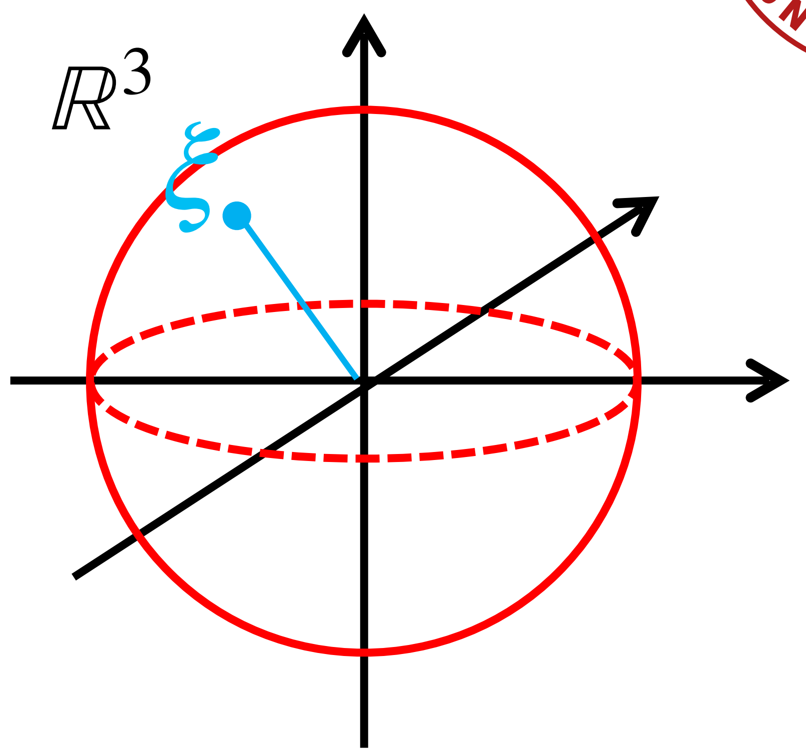
If the point is reachable, any point in that direction is reachable



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Equivalences

- The system is controllable
  - iff $\text{rank}(\mathbb{C}) = n$
- You can choose K to arbitrarily place the eigenvalues of your closed loop system
  - $\dot{x} = (A + BK)x$ **>>K = scipy.signal.place_poles(A, B, *poles*)**
- You can reach anywhere in $\mathbb{R}^n$ in a finite amount of time and energy
  - $\mathscr{R}_t = \mathbb{R}^n$

Controllability Gramians

Controllability Gramians

- We can test if the system is controllable
- … but not how easy it is to control
- … or which directions are the easiest
- … or how we could best improve our control authority

Controllability Gramians

$$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n$$

$$\mathbb{C} = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$$

```
>>rank(ctrb(A,B))
>>[U, S, V] = svd(ℂ, 'econ')
```

- $x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$
- Controllability Gramian
  - $W_t = \int_0^t e^{A\tau}BB^Te^{A^T\tau}d\tau \qquad W_t \in \mathbb{R}^{n\times n}$
  - Discrete time
    - $W_t \approx \mathbb{C}\mathbb{C}^T$
    - $W_t\xi = \lambda\xi$

The SVD of A takes the form: $A = U\Sigma V^T$
U = left singular vector
V = right singular vector
Σ = diagonal matrix of singular values

The eigenvectors with the biggest eigenvalues of the controllability gramian are also the most controllable directions in state space!

Controllability Gramians

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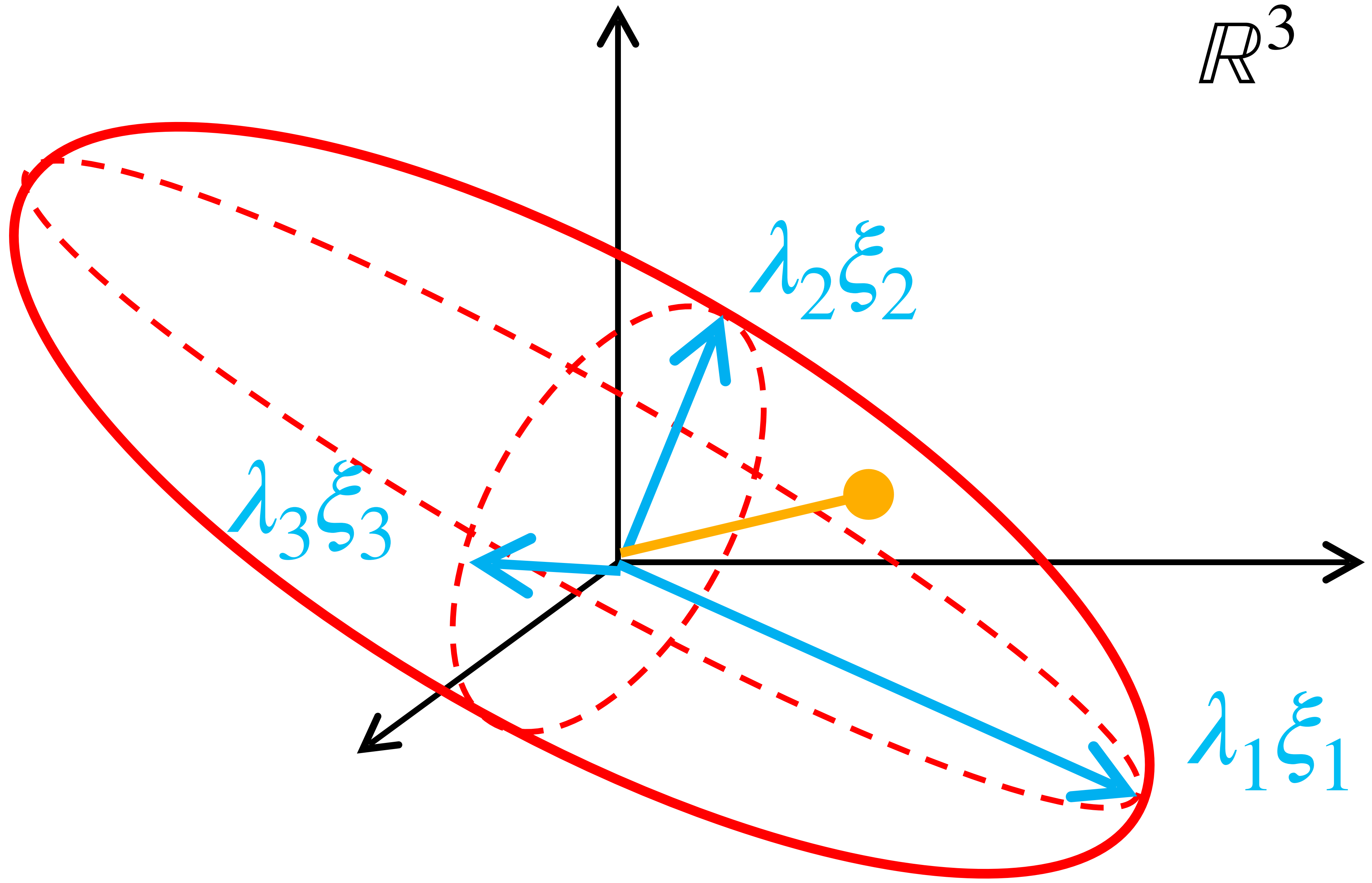

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- Controllability for very high dimensional systems?
- Many directions in $\mathbb{R}^n$ are extremely stable - you only need to control directions that impact your control objective
- Stabilizability

Controllability Gramians

- $x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$
  (convolution of e^{At} with $u(\tau)$)
- Controllability Gramian
  - $W_t = \int_0^t e^{A\tau}BB^Te^{A^T\tau}d\tau \qquad W_t \in \mathbb{R}^{n\times n}$
  - $W_t \approx \mathbb{C}\mathbb{C}^T$
  - $W_t\xi = \lambda\xi$
- Stabilizability
  - A system is stabilizable iff all unstable eigenvectors of A are in the controllable subspace

… and lightly damped (v, inserted before "eigenvectors")

$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n$

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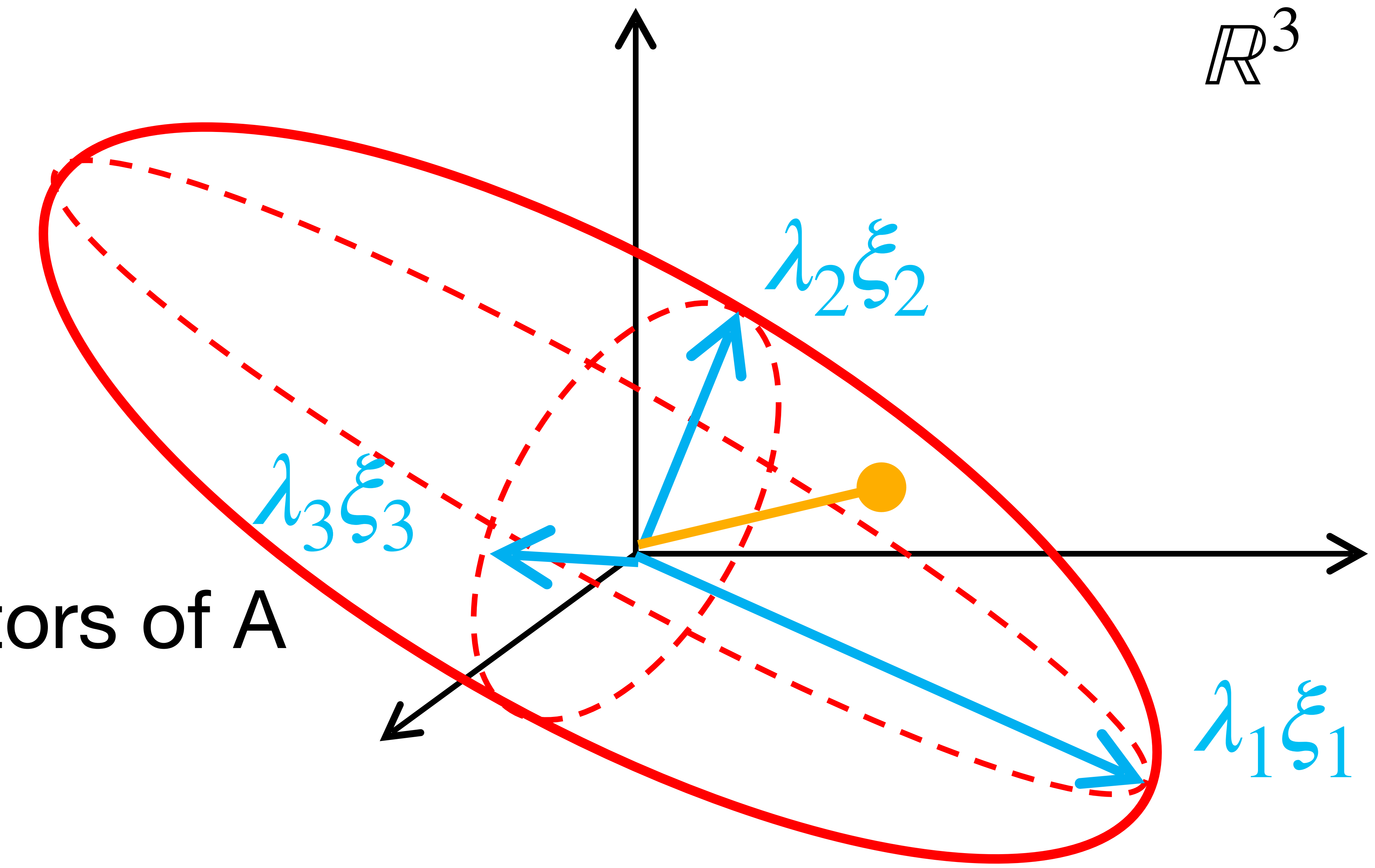

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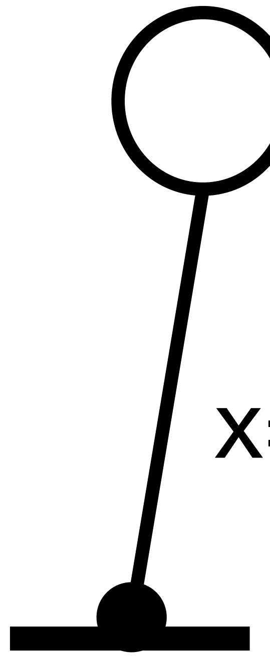
- Observability

$$\dot{x} = Ax + Bu$$



$$x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$

These should look familiar from:

- MATH2940 Linear Algebra
- ECE3250 Signals and Systems
- ECE5210 Theory of Linear Systems
- MAE3260 System Dynamics
- and many others…

Based on "Control Bootcamp", Steve Brunton, UW
https://www.youtube.com/watch?v=Pi7l8mMjYVE