Controllability, Part 2

Fast Robots, ECE4160/5160, MAE 4190/5190

E. Farrell Helbling, 2/27/25

Class Action Items

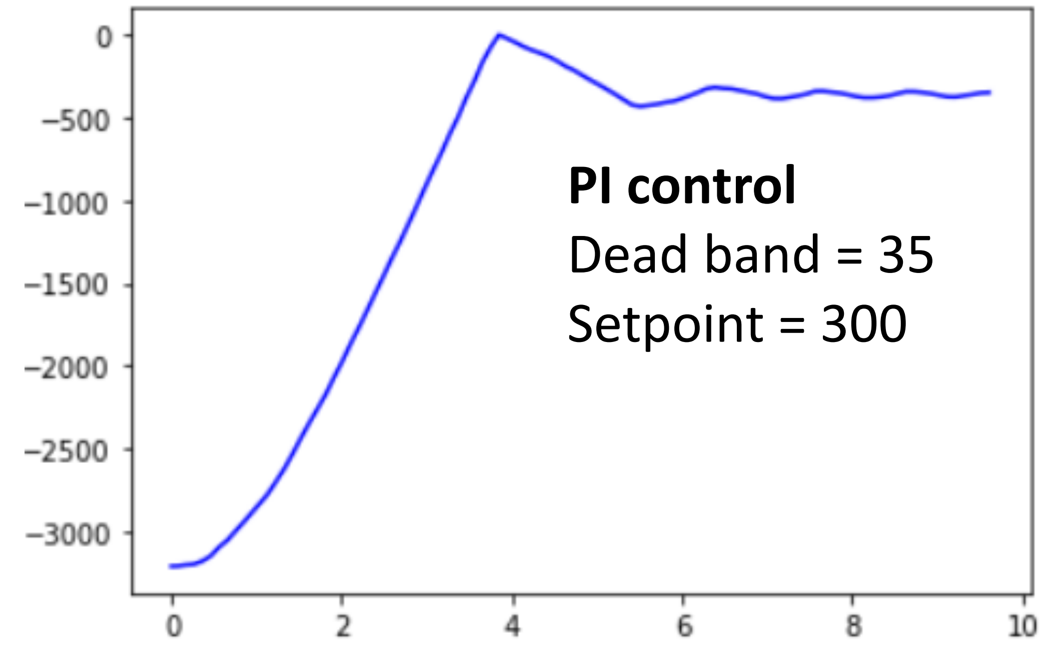
- I am out of town at the beginning of next week.
  - Prof. Petersen is going to fill in to teach class on Tuesday.
  - I will miss lab sections, but will hold additional open hours when I return!
- Lab 4 check-in
- Lab 5 things to consider

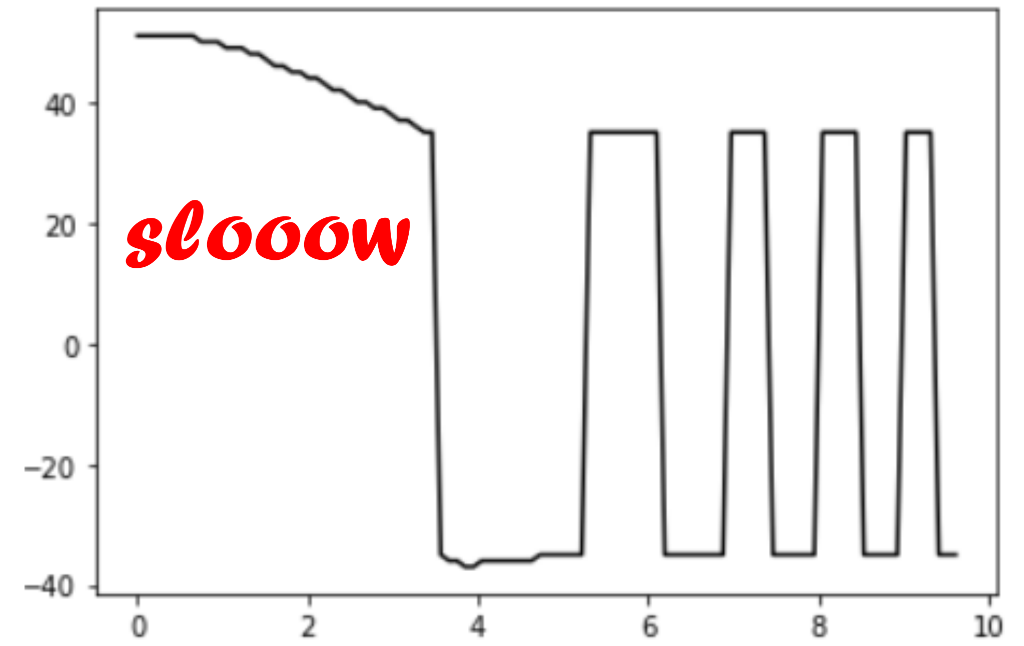
Lab 5

Linear PID

- Great example from last year: https://fast.synthghost.com/lab-5-linear-pid-control/ from Stephan Wagner. You can breeze past his program organization and just get to the lab tasks. Mikayla also had a good report from last year.

    - My advice, go really slowly to start





Linear Systems

Linear Systems — where are we?

- Linear systems review
- Eigenvectors and eigenvalues
- Stability
- Discrete time systems
- Linearizing nonlinear systems
- Controllability
- LQR control
- Observability

$$\dot{x} = Ax + Bu$$

These should look familiar from:

- MATH2940 Linear Algebra
- ECE3250 Signals and Systems
- ECE5210 Theory of Linear Systems
- MAE3260 System Dynamics
- and many others…

Based on "Control Bootcamp", Steve Brunton, UW
https://www.youtube.com/watch?v=Pi7l8mMjYVE

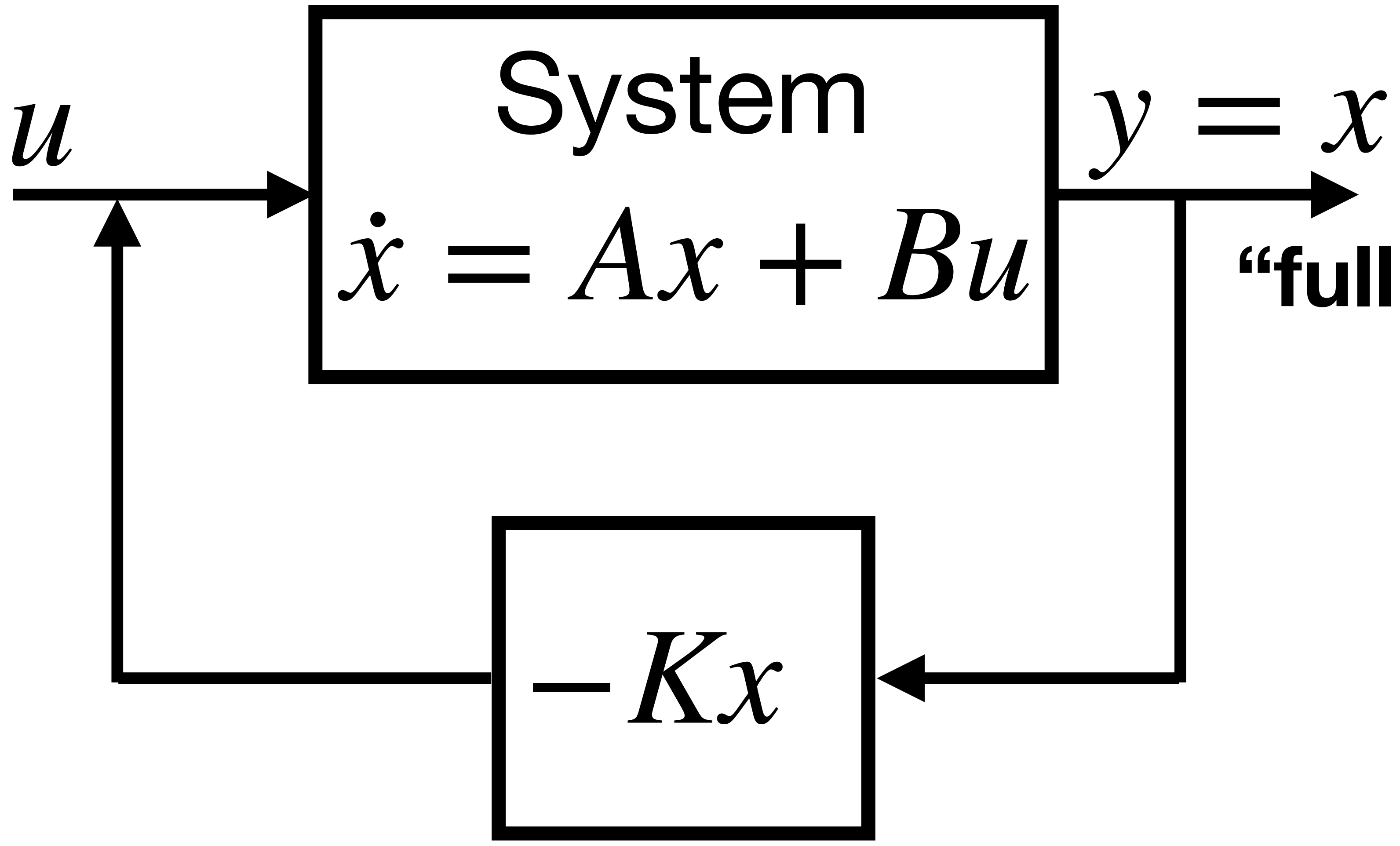
Controllability

Controllability

$$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n$$
$$\dot{x} = Ax - BKx \qquad A \in \mathbb{R}^{n \times n}$$
$$\dot{x} = (A - BK)x \qquad u \in \mathbb{R}^q$$

New dynamics $\qquad B \in \mathbb{R}^{n \times q}$

- Is the system controllable?
  - A system is controllable if you can steer your state x anywhere you want in $\mathbb{R}^n$
  - Matlab `>>rank(ctrb(A,B))`
- How do we design the control law, u?



"full state feedback"

A linear controller (K matrix) can be optimal for linear systems!

Controllability

- Can you control this system?
  - $$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u$$ **Uncontrollable**
  - $$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$ **Controllable**
  - $$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}1 & 1\\ 0 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u$$ **Controllable**
    - Systems with tightly coupled dynamics can be controllable..
    - Get away with using a simple B and fewer sensors

Controllability

- Can you control this system?
  - $$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u$$ **Uncontrollable**
  - $$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$ **Controllable**
  - $$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}1 & 1\\ 0 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u$$ **Controllable**
- Controllability matrix
  - Matlab `>>ctrb(A,B)`
  - $\mathbb{C} = \begin{bmatrix}B & AB & A^2B & \dots & A^{n-1}B\end{bmatrix}$
  - The system is controllable iff $\text{rank}(\mathbb{C}) = n$

For the first system:

$$\mathbb{C} = \begin{bmatrix}0 & 1\cdot 0 + 0\cdot 1\\ 1 & 0\cdot 0 + 2\cdot 1\end{bmatrix} = \begin{bmatrix}0 & 0\\ 1 & 2\end{bmatrix}$$

n = 2, rank = 1

For the second system:

$$\mathbb{C} = \begin{bmatrix}0 & 1 & 1 & 0\\ 1 & 0 & 0 & 2\end{bmatrix}$$

n = 2, rank = 2

For the third system:

$$\mathbb{C} = \begin{bmatrix}0 & 1\cdot 0 + 1\cdot 1\\ 1 & 0\cdot 0 + 2\cdot 1\end{bmatrix} = \begin{bmatrix}0 & 1\\ 1 & 2\end{bmatrix}$$

n = 2, rank = 2

Controllability

$$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n$$

$$\dot{x} = Ax - BKx \qquad A \in \mathbb{R}^{n \times n}$$

$$\dot{x} = (A - BK)x \qquad u \in \mathbb{R}^q$$

New dynamics $\quad B \in \mathbb{R}^{n \times q}$

- Can you control this system?
  - $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$
  - $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$
  - $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$
- Controllability matrix
  - Matlab `>>ctrb(A,B)`
  - $\mathbb{C} = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$
  - The system is controllable iff $\text{rank}(\mathbb{C}) = n$

FYI! Just because a linearized, nonlinear system is uncontrollable, this does not mean that the nonlinear system is uncontrollable!

Controllability in Discrete Time

$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n$

$\mathbb{C} = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$

- Why does $\mathbb{C}$ predict controllability?
- Discrete time impulse response: $x(k+1) = \tilde{A}x(k) + \tilde{B}u(k)$

(assume a single input actuator)

$u(0) = 1 \qquad x(0) = 0$

$u(1) = 0 \qquad x(1) = \tilde{A}x(0) + \tilde{B}u(0) = \tilde{B}$

$u(2) = 0 \qquad x(2) = \tilde{A}x(1) + \tilde{B}u(1) = \tilde{A}\tilde{B}$

$u(3) = 0 \qquad x(3) = \tilde{A}^2\tilde{B}$

$\vdots \qquad \vdots$

$u(m) = 0 \qquad x(m) = \tilde{A}^{m-1}\tilde{B}$

If the system is controllable, then the impulse response affects every state in $\mathbb{R}^n$

Review

- Linear system: $\dot{x} = Ax$
- Solution: $x(t) = e^{At}x(0)$
- Eigenvectors: $T = \begin{bmatrix} \xi_1 & \xi_2 & \dots & \xi_n \end{bmatrix}$
- Eigenvalues: $D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$

```
>>[T,D] = eig(A)
```

- Linear Transform: $AT = TD$
- Solution: $e^{At} = e^{TDT^{-1}t}$
- Mapping from x to z to x: $x(t) = Te^{Dt}T^{-1}x(0)$
- Stability in continuous time: $\lambda = a + ib$, stable iff $a < 0$
- Discrete time: $x(k+1) = \tilde{A}x(k)$, where $\tilde{A} = e^{A\Delta t}$
- Stability in discrete time: $\tilde{\lambda}^n = R^n e^{in\theta}$, stable iff $R < 1$
- Nonlinear systems: $\dot{x} = f(x)$
- Linearization: $\left.\frac{Df}{Dx}\right|_{\bar{x}}$
- Controllability: $\dot{x} = (A - BK)x$

```
>>rank(ctrb(A,B))
```

Reachability

Controllability and Reachability

$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n$

$\mathbb{C} = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$

Reachability

- $\mathcal{R}_t$: states that are reachable at time t
- $\mathcal{R}_t = \{\xi \in \mathbb{R}^n$ for which there is an input $u(t)$ that makes $x(t) = \xi$

Equivalences

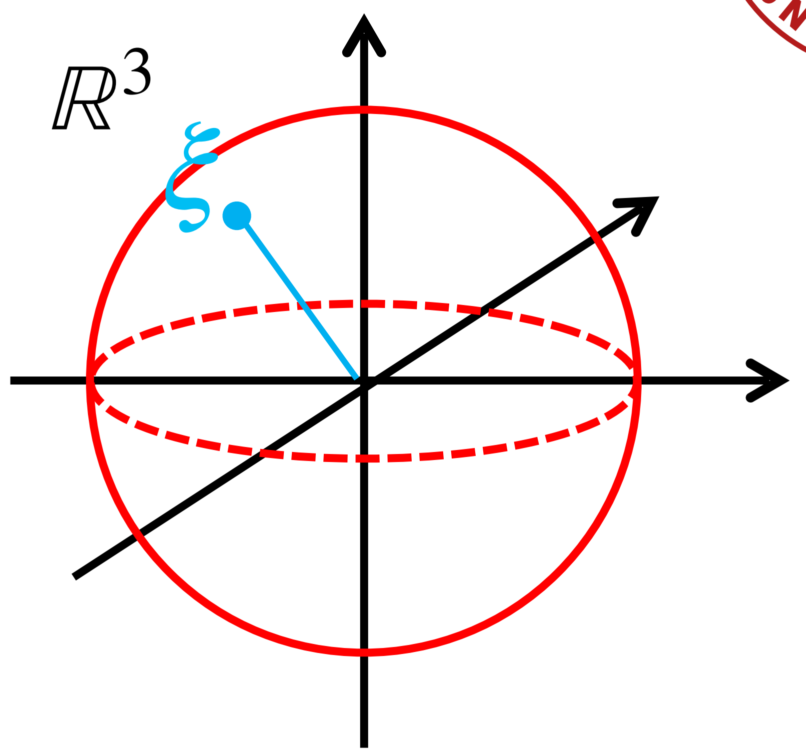
- The system is controllable
  - iff $\text{rank}(\mathbb{C}) = n$
- You can choose K to arbitrarily place the eigenvalues of your closed loop system
  - $\dot{x} = (A - BK)x$
- You can reach anywhere in $\mathbb{R}^n$ in a finite amount of time and energy
  - $\mathcal{R}_t = \mathbb{R}^n$

Controllability and Reachability

$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n$

$\mathbb{C} = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$

If the point is reachable, any point in that direction is reachable



Reachability

- $\mathscr{R}_t$: states that are reachable at time t
- $\mathscr{R}_t = \{\xi \in \mathbb{R}^n$ for which there is an input $u(t)$ that makes $x(t) = \xi$

Equivalences

- The system is controllable
  - iff $\text{rank}(\mathbb{C}) = n$
- You can choose K to arbitrarily place the eigenvalues of your closed loop system
  - $\dot{x} = (A - BK)x$

```
>>K = scipy.signal.place_poles(A, B, poles)
```

- You can reach anywhere in $\mathbb{R}^n$ in a finite amount of time and energy
  - $\mathscr{R}_t = \mathbb{R}^n$

Controllability Gramians

Controllability Gramians

- We can test if the system is controllable
- … but not how easy it is to control
- … or which directions are the easiest
- … or how we could best improve our control authority

Controllability Gramians

- $x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$
- Controllability Gramian
  - $W_t = \int_0^t e^{A\tau}BB^Te^{A^T\tau}d\tau \qquad W_t \in \mathbb{R}^{n\times n}$
  - $W_t\xi = \lambda\xi$
  - Discrete time
    - $W_t \approx \mathbb{C}\mathbb{C}^T$

$$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n$$

$$\mathbb{C} = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$$

```
>>rank(ctrb(A,B))

>>[U, S, V] = svd(ℂ, 'econ')
```

The SVD of A takes the form: $A = U\Sigma V^T$
U = left singular vector
V = right singular vector
Σ = diagonal matrix of singular values

The eigenvectors with the biggest eigenvalues of the controllability gramian are also the most controllable directions in state space!

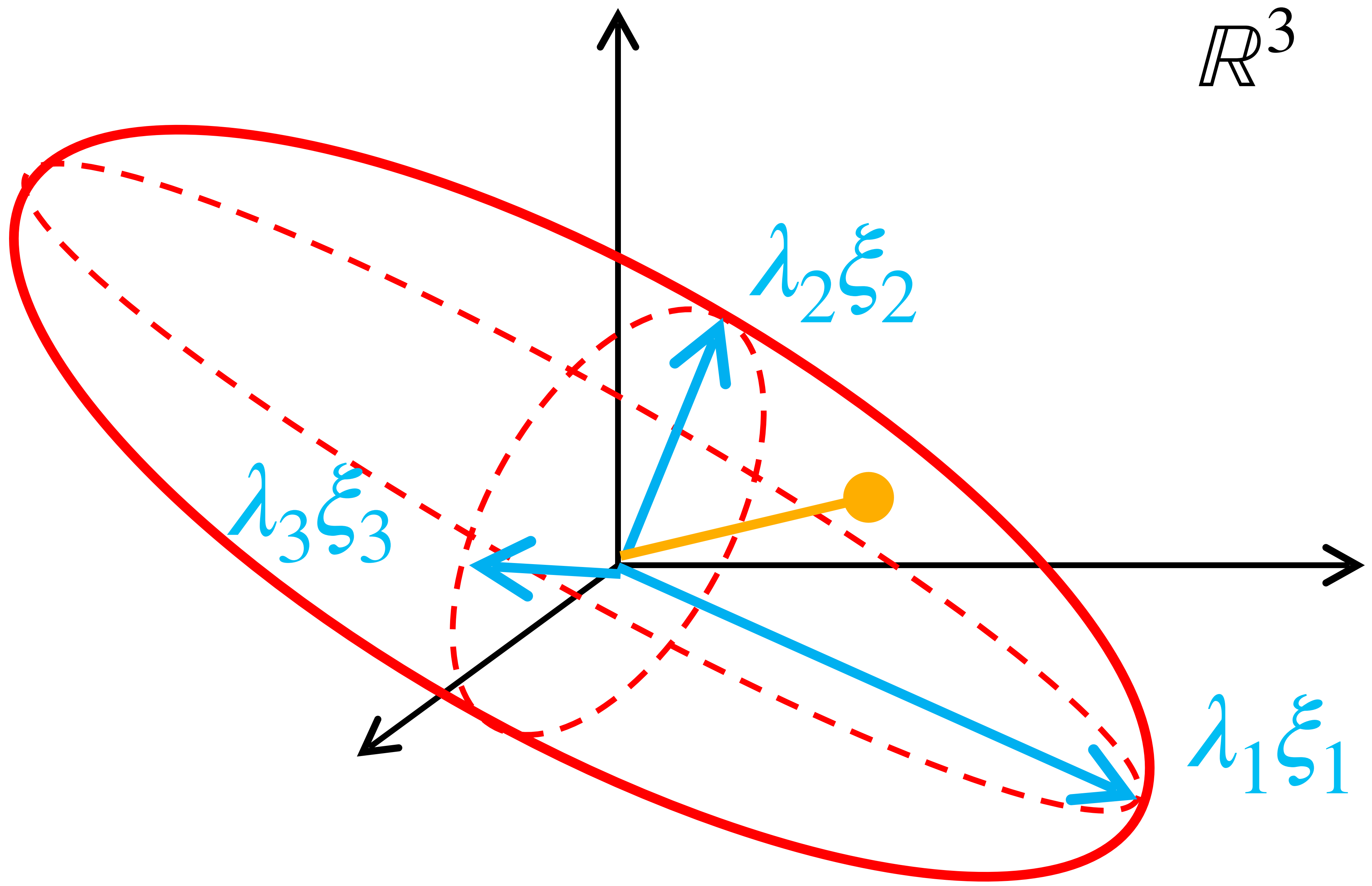
Controllability Gramians

$$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n$$

$$\mathbb{C} = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$$

```
>>rank(ctrb(A,B))
>>[U, S, V] = svd(ℂ, ‘econ’)
```

- $x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$
- Controllability Gramian
  - $W_t = \int_0^t e^{A\tau}BB^T e^{A^T\tau}d\tau \qquad W_t \in \mathbb{R}^{n\times n}$
  - $W_t\xi = \lambda\xi$
  - Discrete time
    - $W_t \approx \mathbb{C}\mathbb{C}^T$



Controllability Gramians

$$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n$$

$$\mathbb{C} = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$$

```
>>rank(ctrb(A,B))
>>[U, S, V] = svd(ℂ, 'econ')
```

- Controllability for very high dimensional systems?
- Many directions in $\mathbb{R}^n$ are extremely stable - you only need to control directions that impact your control objective
- Stabilizability

Controllability Gramians

- $x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$

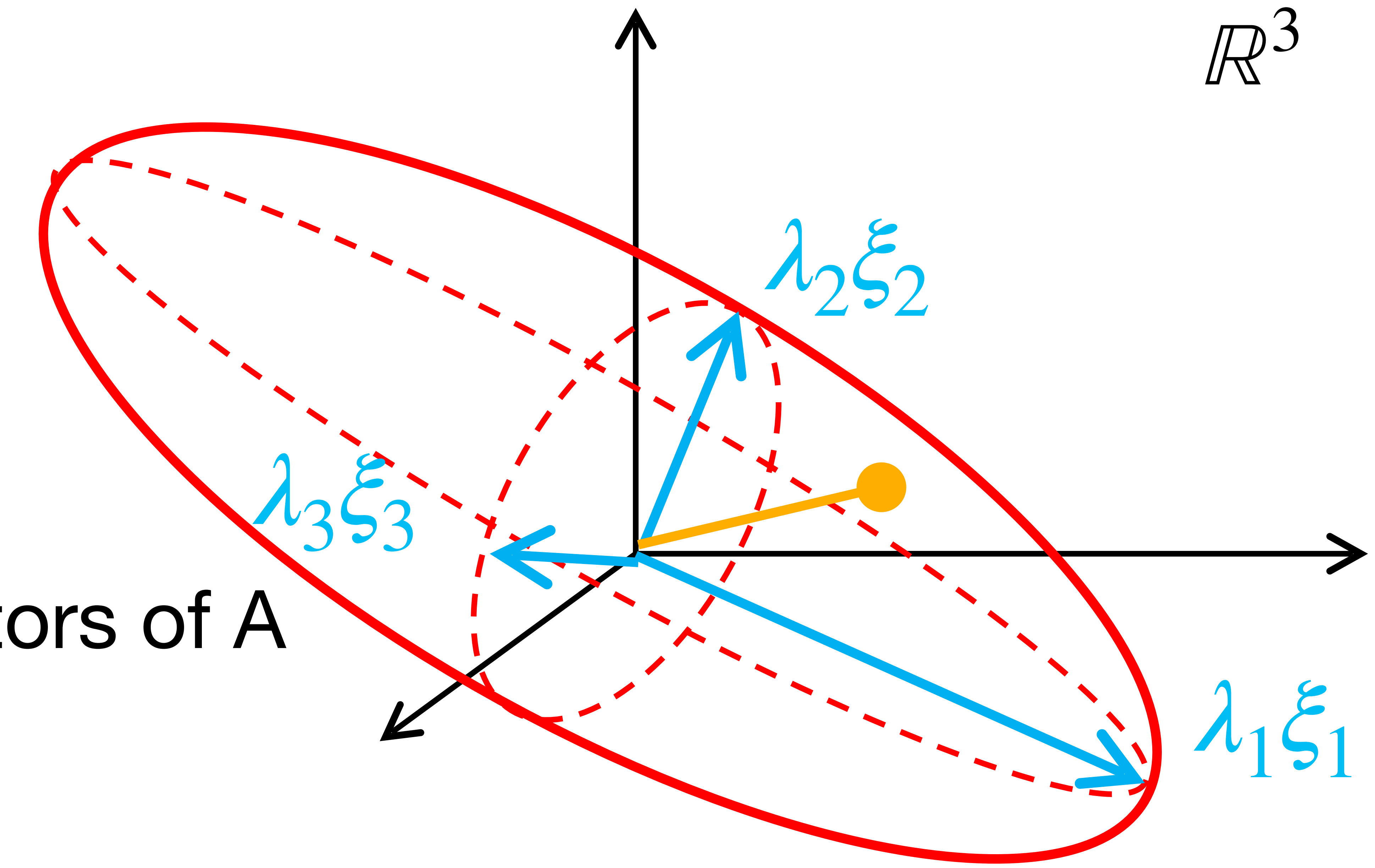
  (convolution of e^{At} with $u(\tau)$)
- Controllability Gramian
  - $W_t = \int_0^t e^{A\tau}BB^T e^{A^T\tau}d\tau \qquad W_t \in \mathbb{R}^{n\times n}$
  - $W_t \approx \mathbb{C}\mathbb{C}^T$
  - $W_t\xi = \lambda\xi$
- Stabilizability
  - A system is stabilizable iff all unstable eigenvectors of A are in the controllable subspace

  … and lightly damped (inserted before "eigenvectors")

$\dot{x} = Ax + Bu \qquad x \in \mathbb{R}^n$

$\mathbb{C} = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$

```
>>rank(ctrb(A,B))
>>[U, S, V] = svd(ℂ, 'econ')
```



Review

- Linear system: $\dot{x} = Ax$
- Solution: $x(t) = e^{At}x(0)$
- Eigenvectors: $T = \begin{bmatrix} \xi_1 & \xi_2 & \dots & \xi_n \end{bmatrix}$
- Eigenvalues: $D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$

```
>>[T,D] = eig(A)
```

- Linear Transform: $AT = TD$
- Solution: $e^{At} = e^{TDT^{-1}t}$
- Mapping from x to z to x: $x(t) = Te^{Dt}T^{-1}x(0)$
- Stability in continuous time: $\lambda = a + ib$, stable iff $a < 0$
- Discrete time: $x(k+1) = \tilde{A}x(k)$, where $\tilde{A} = e^{A\Delta t}$
- Stability in discrete time: $\tilde{\lambda}^n = R^n e^{in\theta}$, stable iff $R < 1$
- Nonlinear systems: $\dot{x} = f(x)$
- Linearization: $\left.\frac{Df}{Dx}\right|_{\bar{x}}$
- Controllability: $\dot{x} = (A - BK)x$

```
>>rank(ctrb(A,B))
```

- Reachability
- Controllability Gramian

Linear Systems — where are we?

- Linear systems review
- Eigenvectors and eigenvalues
- Stability
- Discrete time systems
- Linearizing nonlinear systems
- Controllability
- LQR control
- Observability

$$\dot{x} = Ax + Bu$$

These should look familiar from:

- MATH2940 Linear Algebra
- ECE3250 Signals and Systems
- ECE5210 Theory of Linear Systems
- MAE3260 System Dynamics
- and many others…

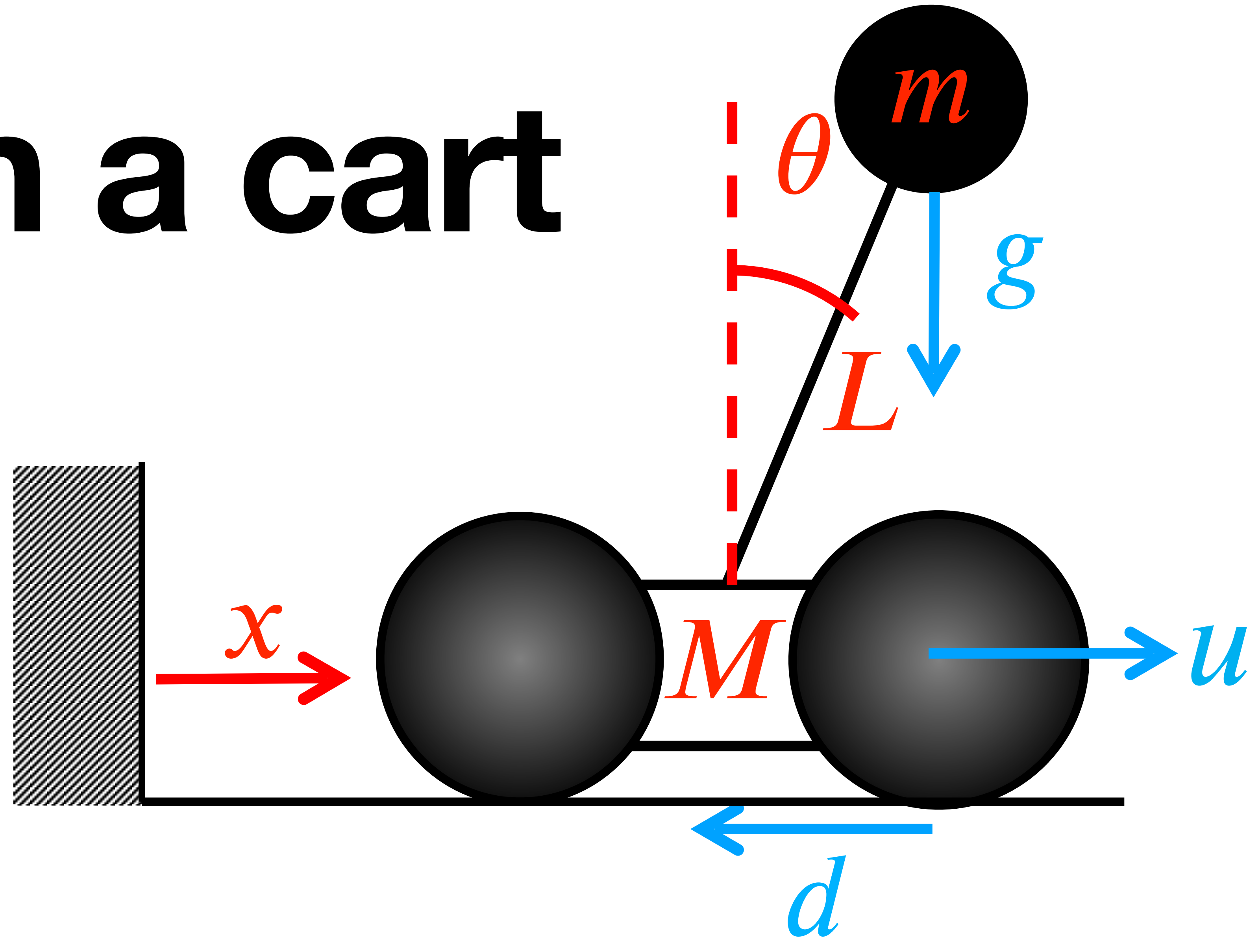
Based on “Control Bootcamp”, Steve Brunton, UW
https://www.youtube.com/watch?v=Pi7l8mMjYVE

Cart Pole

Based entirely on Steve Brunton's Controlled Bootcamp Lecture Series

Inverted pendulum on a cart

How do we reason about this system?



Force acting on the cart in the x direction

1. Eqs. of motion ↓

2. State space model → 3. Fixed points → 4. Jacobian → 5. Is it controllable? ↓

6. Add linear control

$$y = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix}$$

$$\bar{y} = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}$$

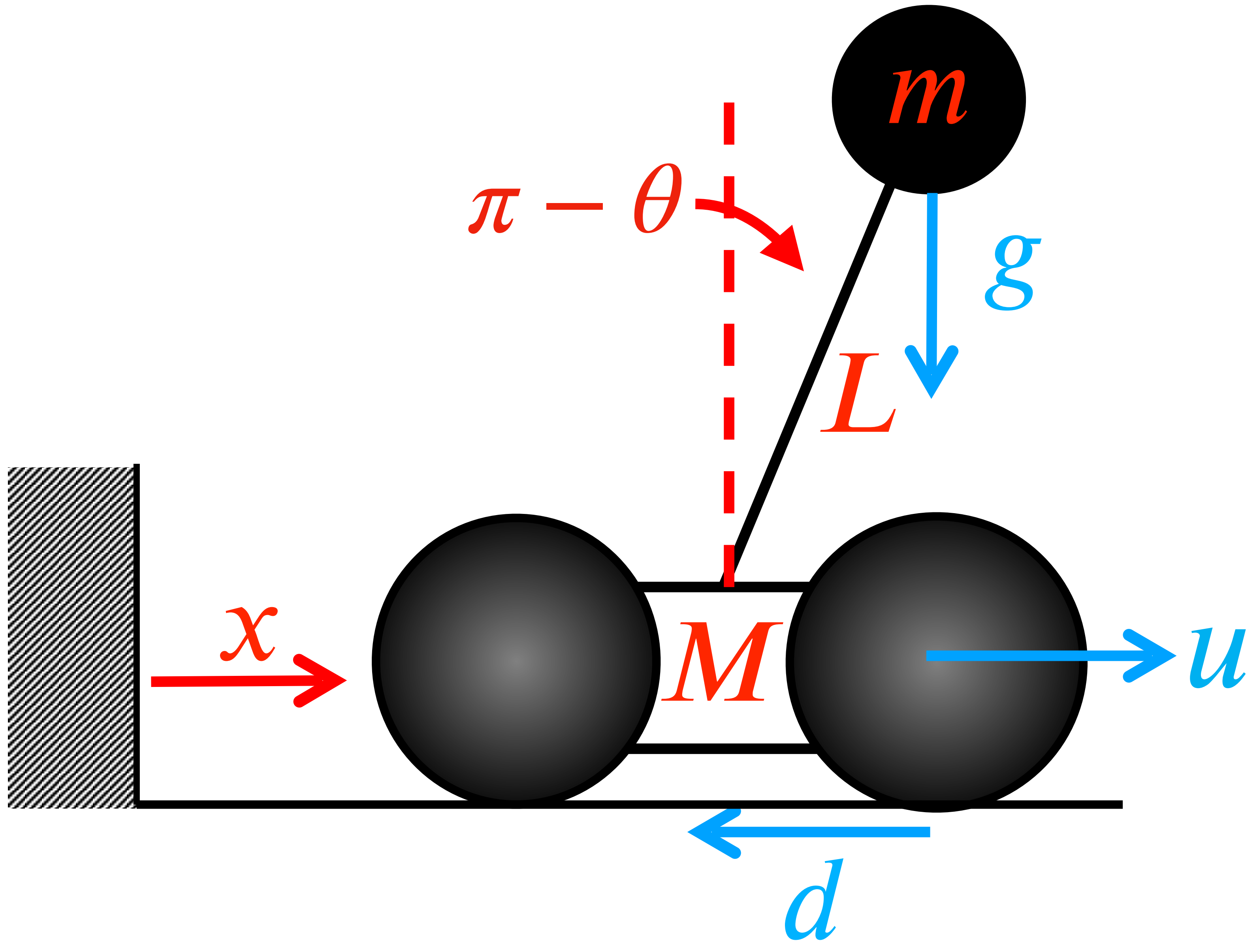
$$\left. \frac{Df}{Dy} \right|_{\bar{y}}$$

$$\dot{y} = Ay + Bu$$

$$\dot{y} = (A - BK)y$$

Inverted pendulum on a cart

Equations of motion



Euler Lagrange Formulation

$$\frac{d}{dt}\left(\frac{\delta L}{\delta \dot{q}_i}\right) - \frac{\delta L}{\delta q_i} = Q_i \qquad L = T - V$$

$$V = mgy_m = -mgl\cos(\theta)$$

$$x_m = x + l\sin(\theta)$$

$$\dot{x}_m = \dot{x} + l\dot{\theta}\cos(\theta)$$

$$y_m = -l\cos(\theta)$$

$$\dot{y}_m = l\dot{\theta}\sin(\theta)$$

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\dot{x}_m^2 + \frac{1}{2}m\dot{y}_m^2$$

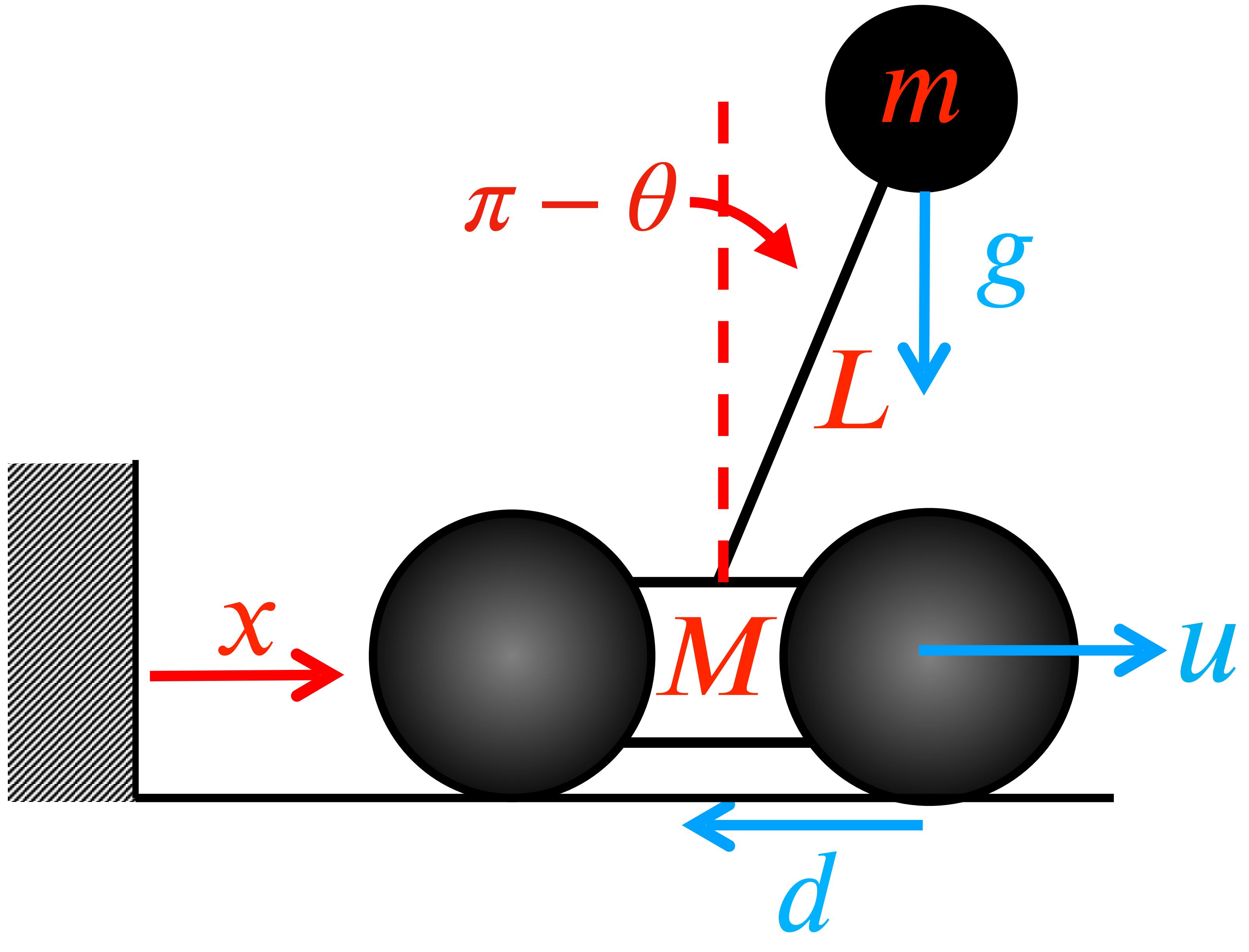
$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}_m^2 + \dot{y}_m^2)$$

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + 2l\dot{\theta}\dot{x}\cos(\theta) + l^2\dot{\theta}^2\cos^2(\theta) + l^2\dot{\theta}^2\sin^2(\theta))$$

$$T = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}ml^2\dot{\theta}^2 + ml\dot{\theta}\dot{x}\cos(\theta)$$

Inverted pendulum on a cart

Equations of motion in x



Euler Lagrange Formulation

$$\frac{d}{dt}\left(\frac{\delta L}{\delta \dot{q}_i}\right) - \frac{\delta L}{\delta q_i} = Q_i$$

$$L = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}ml^2\dot{\theta}^2 + ml\dot{\theta}\dot{x}\cos(\theta) + mgl\cos(\theta)$$

$$q_i = x, \quad Q_i = F - d\dot{x}$$

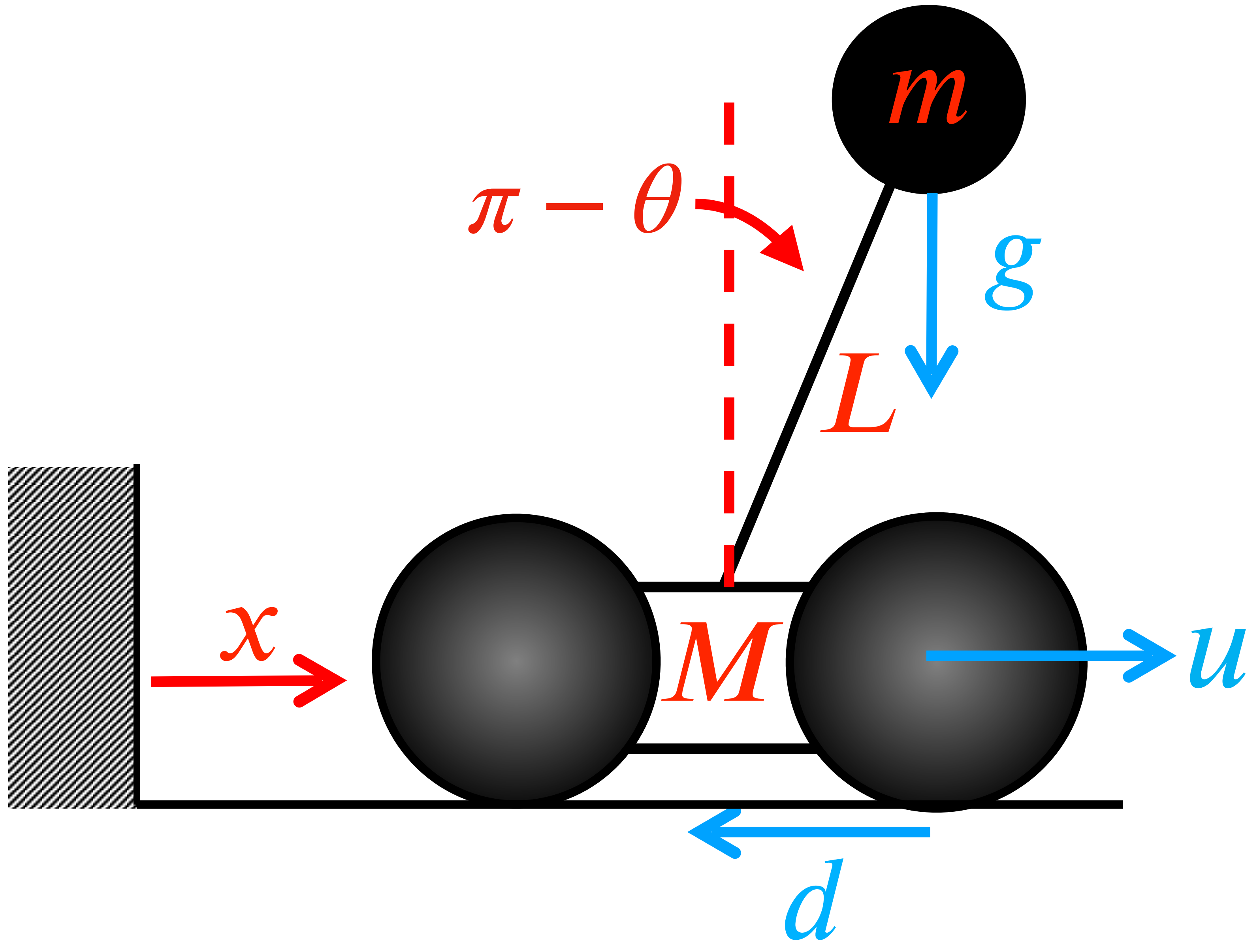
$$\frac{\delta L}{\delta \dot{x}} = (M+m)\dot{x} + ml\dot{\theta}\cos(\theta) \qquad \frac{\delta L}{\delta x} = 0$$

$$\frac{d}{dt}\left(\frac{\delta L}{\delta \dot{x}}\right) = (M+m)\ddot{x} + ml\ddot{\theta}\cos(\theta) - ml\dot{\theta}^2\sin(\theta)$$

$$(M+m)\ddot{x} + ml\ddot{\theta}\cos(\theta) - ml\dot{\theta}^2\sin(\theta) = F - d\dot{x}$$

Inverted pendulum on a cart

Equations of motion in θ



Euler Lagrange Formulation

$$\frac{d}{dt}\left(\frac{\delta L}{\delta \dot{q}_i}\right) - \frac{\delta L}{\delta q_i} = Q_i$$

$$L = \frac{1}{2}(M+m)\dot{x}^2 + \frac{1}{2}ml^2\dot{\theta}^2 + ml\dot{\theta}\dot{x}\cos(\theta) + mgl\cos(\theta)$$

$$q_i = \theta, \quad Q_i = 0$$

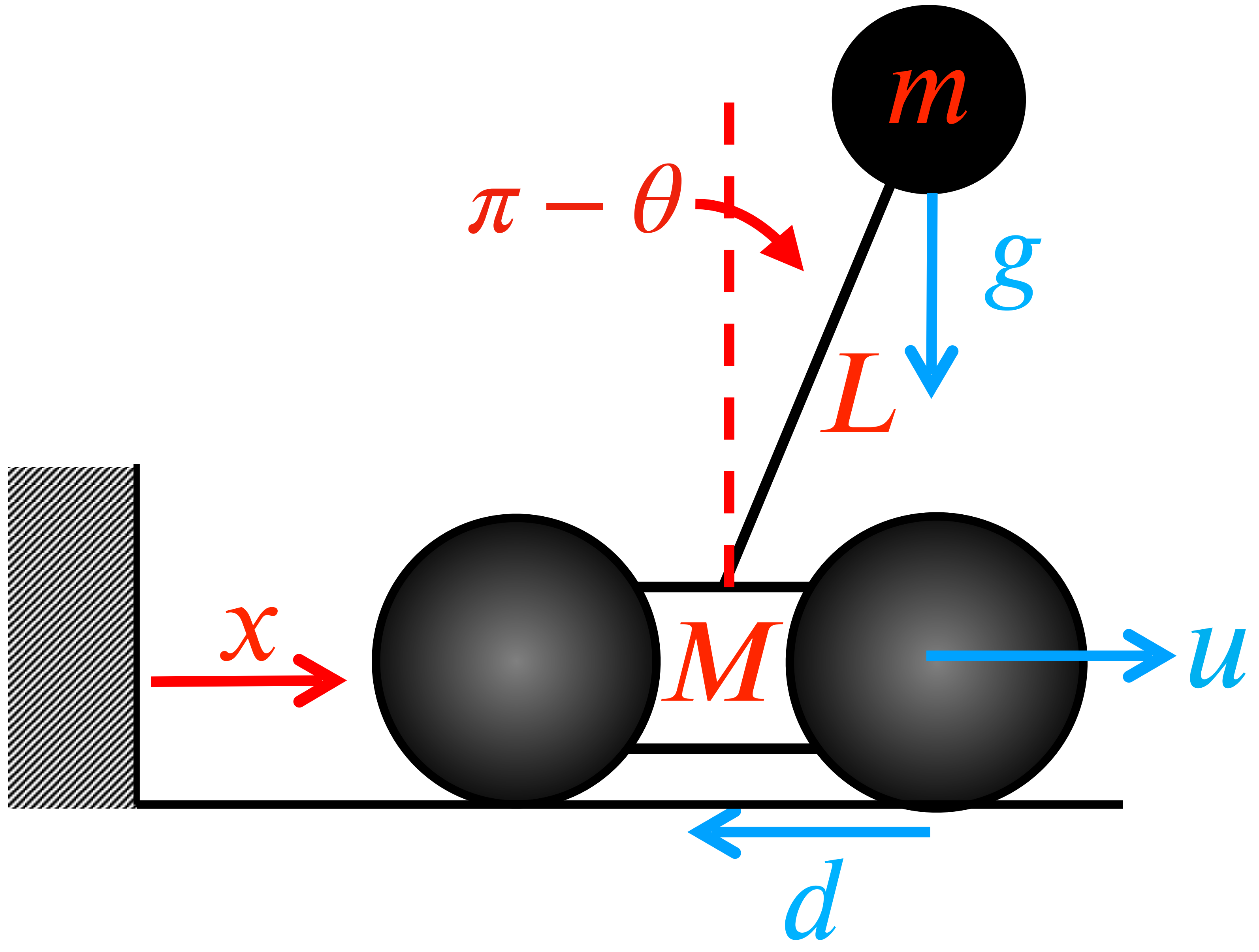
$$\frac{\delta L}{\delta \dot{\theta}} = ml^2\dot{\theta} + ml\dot{x}\cos(\theta) \qquad \frac{\delta L}{\delta \theta} = -ml\dot{\theta}\dot{x}\sin(\theta) - mgl\sin(\theta)$$

$$\frac{d}{dt}\left(\frac{\delta L}{\delta \dot{\theta}}\right) = ml^2\ddot{\theta} + ml\ddot{x}\cos(\theta) - ml\dot{x}\dot{\theta}\sin(\theta)$$

$$ml^2\ddot{\theta} + ml\ddot{x}\cos(\theta) + mgl\sin(\theta) = 0$$

Inverted pendulum on a cart

Equations of motion



$$(M+m)\ddot{x} + ml\ddot{\theta}\cos(\theta) - ml\dot{\theta}^2\sin(\theta) = F - d\dot{x}$$

$$ml^2\ddot{\theta} + ml\ddot{x}\cos(\theta) + mgl\sin(\theta) = 0$$

$$\begin{bmatrix} (M+m) & ml\cos(\theta) \\ ml\cos(\theta) & ml^2 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} F + ml\dot{\theta}^2\sin(\theta) - d\dot{x} \\ -mgl\sin(\theta) \end{bmatrix}$$

$$\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \dfrac{(ml^2)(F + ml\dot{\theta}^2\sin(\theta) - d\dot{x} + mg\cos(\theta)\sin(\theta))}{\Delta} \\ \dfrac{(-ml\cos(\theta))(F + ml\dot{\theta}^2\sin(\theta) - d\dot{x}) - (M+m)mgl\sin(\theta)}{\Delta} \end{bmatrix}$$

$$\det = \Delta = ml^2(M + m(1 - \cos^2(\theta)))$$

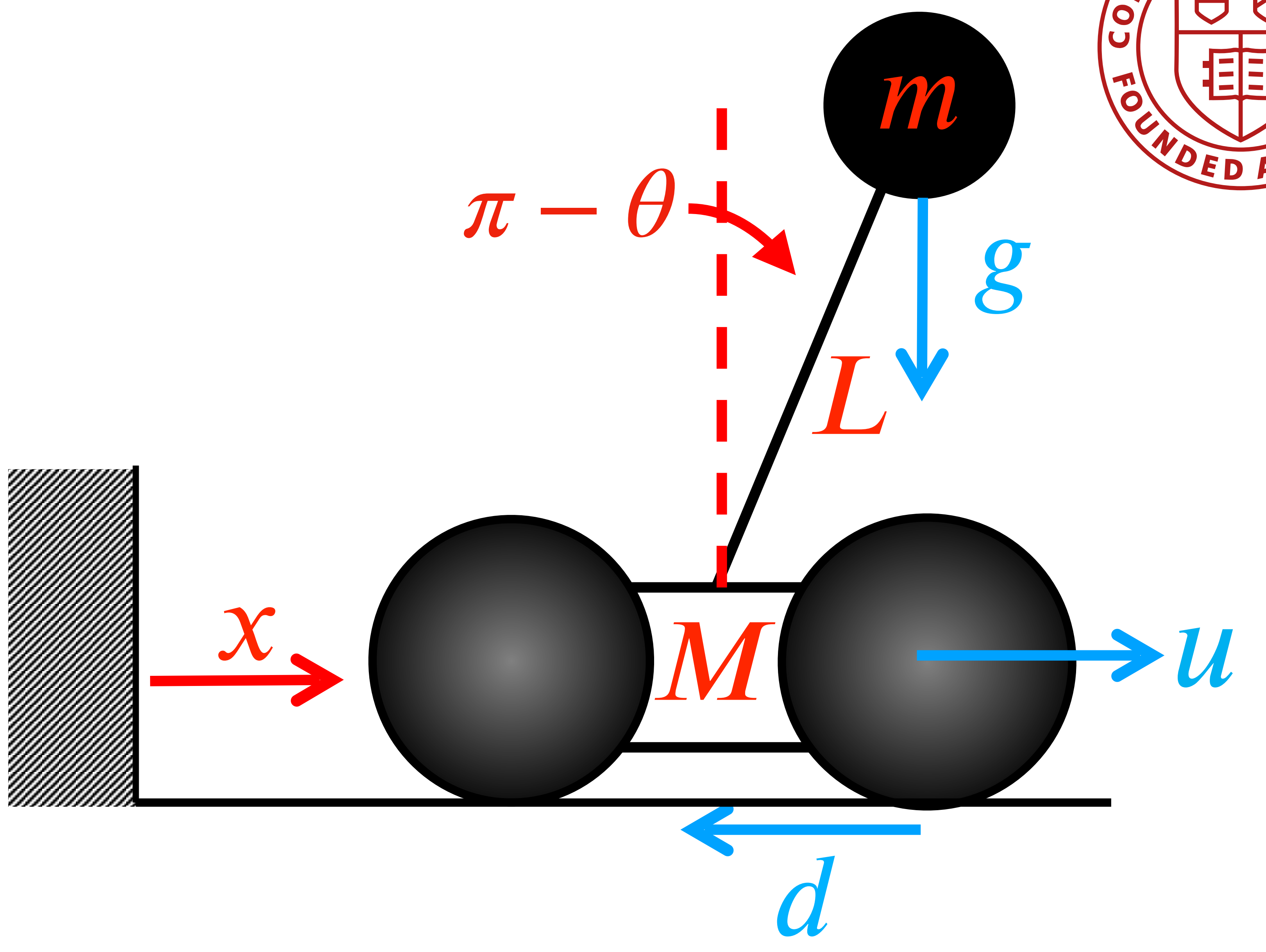
Inverted pendulum on a cart

Linearize the nonlinear system

$$\begin{bmatrix}\ddot{x}\\ \ddot{\theta}\end{bmatrix} = \begin{bmatrix}\frac{(ml^2)(F + ml\dot{\theta}^2\sin(\theta) - d\dot{x} + mg\cos(\theta)\sin(\theta))}{\Delta}\\ \frac{(-ml\cos(\theta))(F + ml\dot{\theta}^2\sin(\theta) - d\dot{x}) - (M+m)mgl\sin(\theta)}{\Delta}\end{bmatrix}$$

$$\Delta = ml^2(M + m(1 - \cos^2(\theta)))$$

$$\frac{d}{dt}\begin{bmatrix}x\\ \dot{x}\\ \theta\\ \dot{\theta}\end{bmatrix} = \begin{bmatrix}\dot{x}\\ \frac{(ml^2)(F + ml\dot{\theta}^2\sin(\theta) - d\dot{x} + mg\cos(\theta)\sin(\theta))}{\Delta}\\ \dot{\theta}\\ \frac{(-ml\cos(\theta))(F + ml\dot{\theta}^2\sin(\theta) - d\dot{x}) - (M+m)mgl\sin(\theta)}{\Delta}\end{bmatrix}$$



Linearize about:
$x = \text{free},\ \dot{x} = 0,\ \theta = \{0, \pi\},\ \dot{\theta} = 0$

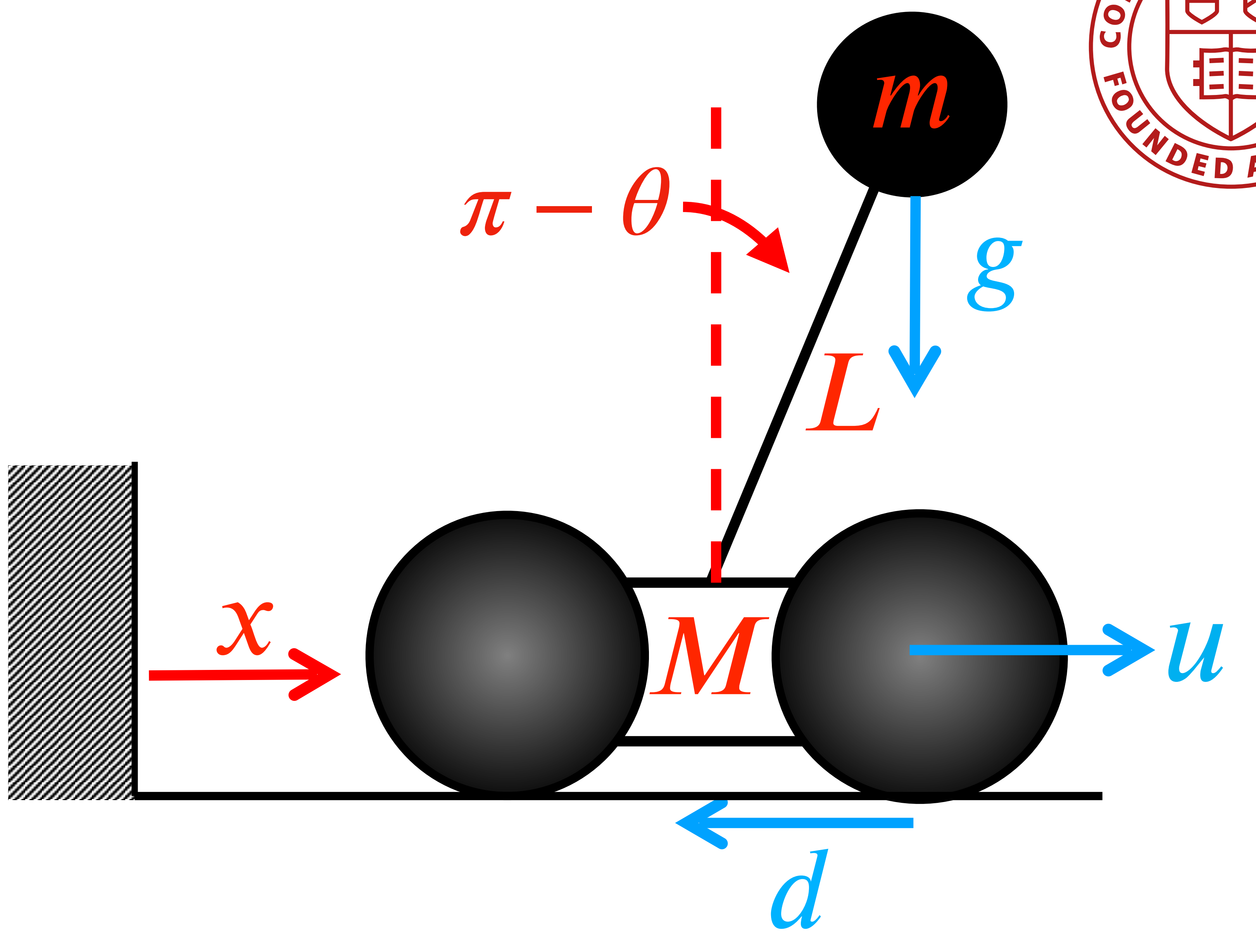
$$\frac{d}{dt}\begin{bmatrix}x\\ \dot{x}\\ \theta\\ \dot{\theta}\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & -\frac{d}{M} & \frac{mg}{M} & 0\\ 0 & 0 & 0 & 1\\ 0 & -\frac{\text{up}\ d}{ML} & \frac{\text{up}(m+M)g}{ML} & 0\end{bmatrix}\begin{bmatrix}x\\ \dot{x}\\ \theta\\ \dot{\theta}\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{M}\\ 0\\ \frac{\text{up}}{ML}\end{bmatrix}F$$

where up = 1 at $\theta = \pi$ and up = -1 at $\theta = 0$

Inverted pendulum on a cart

Eigenvalues, Stability, Controllability

$$\frac{d}{dt}\begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{d}{M} & \frac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{\text{up}\ d}{ML} & \frac{\text{up}(m+M)g}{ML} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{\text{up}}{ML} \end{bmatrix} F$$



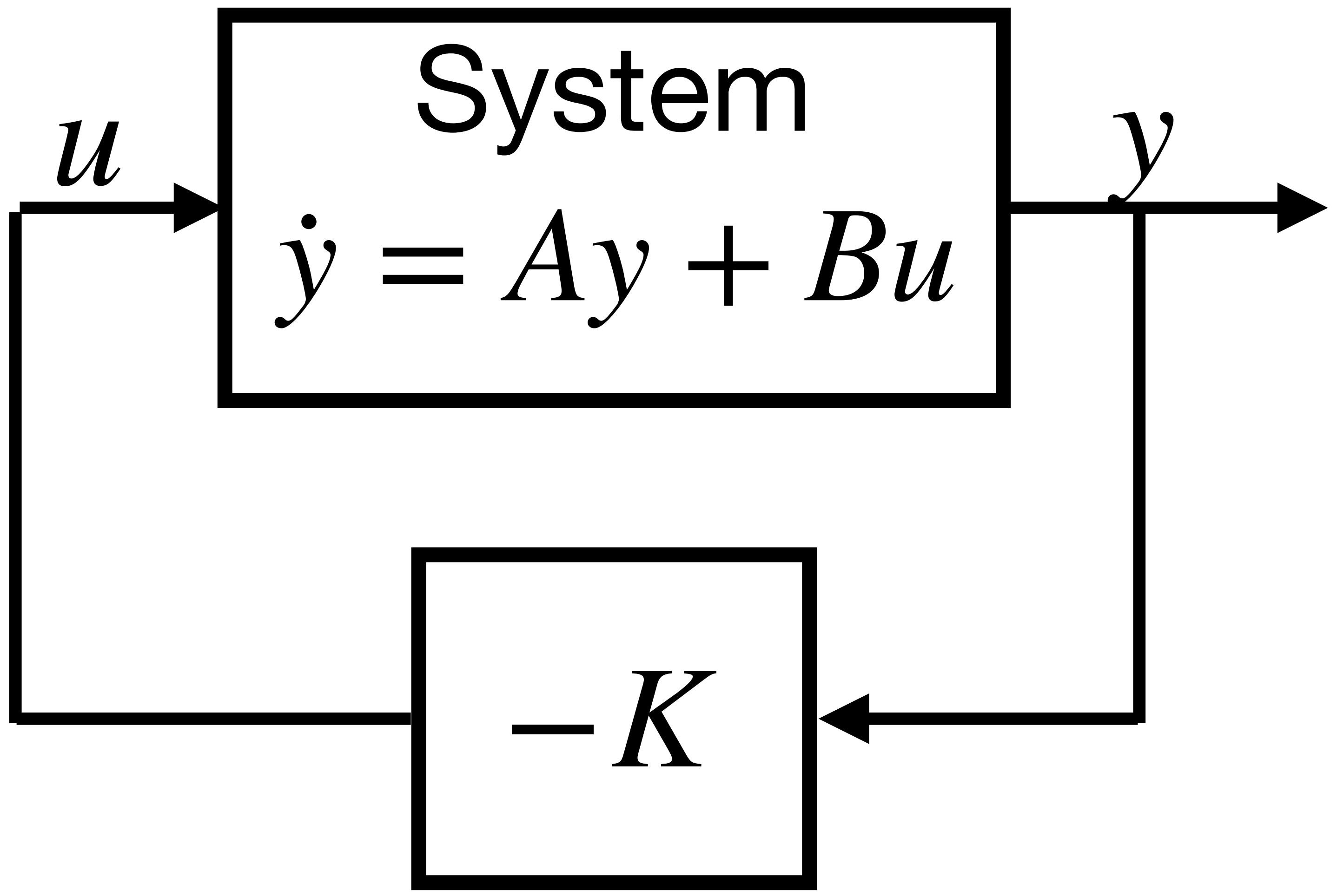
Let's go to Matlab!

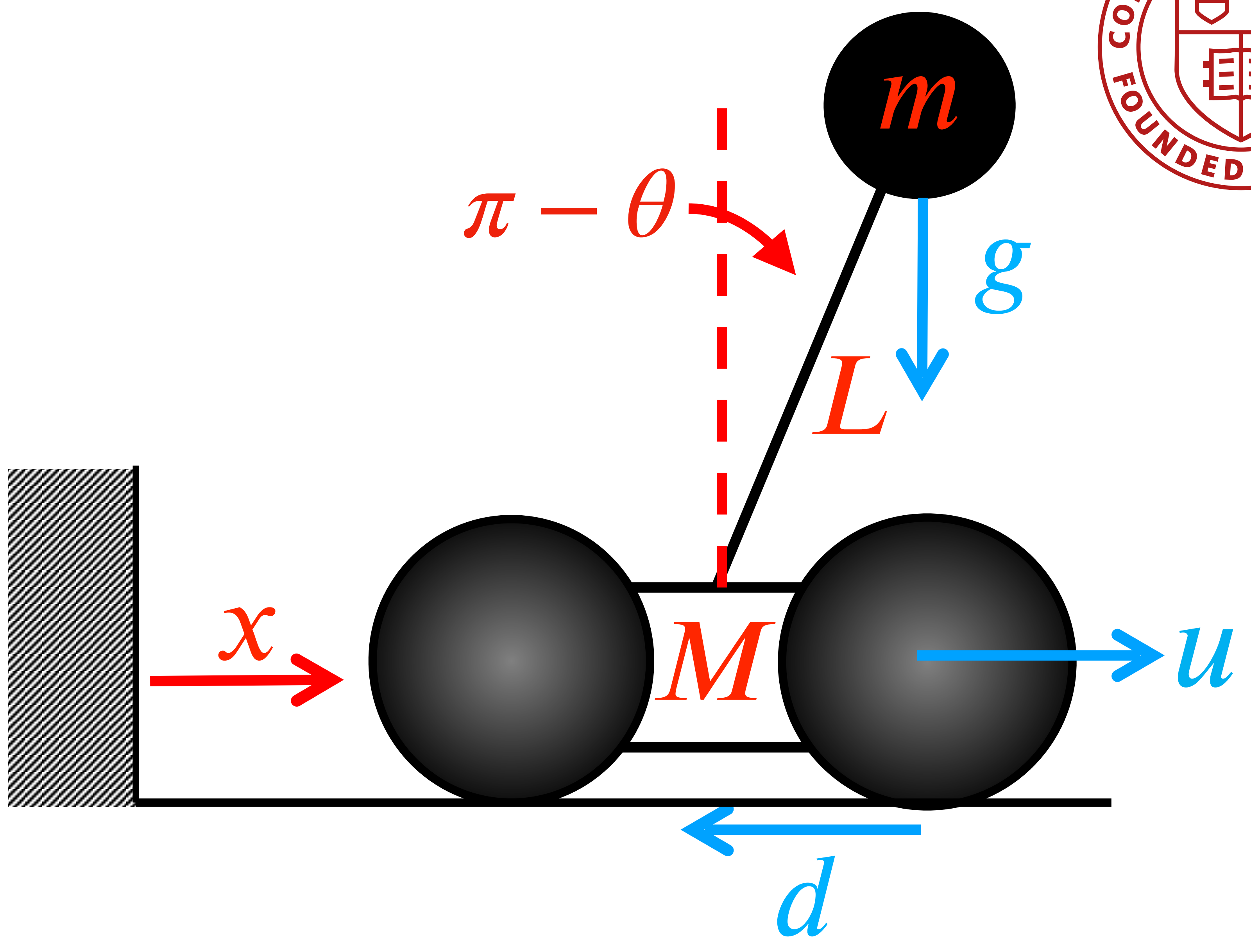
- Check nonlinear equations
- Run open-loop simulation
- Check for stability, controllability

Inverted pendulum on a cart

Control Law

$$\dot{y} = Ay + Bu$$
$$u = -Ky$$
$$\dot{y} = (A - BK)y$$





Let's go to Matlab!

- Pole placement.
  - Define poles `>>eigs = [-1 -1.2 -1.3 -1.4];`
  - K-matrix `>>K = place(A,B,eigs)`

Pole Placement

- Python
  - `K = scipy.signal.place_poles(A,B,poles)`[1]
- Barely stable eigenvalues: not enough control authority
- More negative eigenvalues: faster response, less robust system

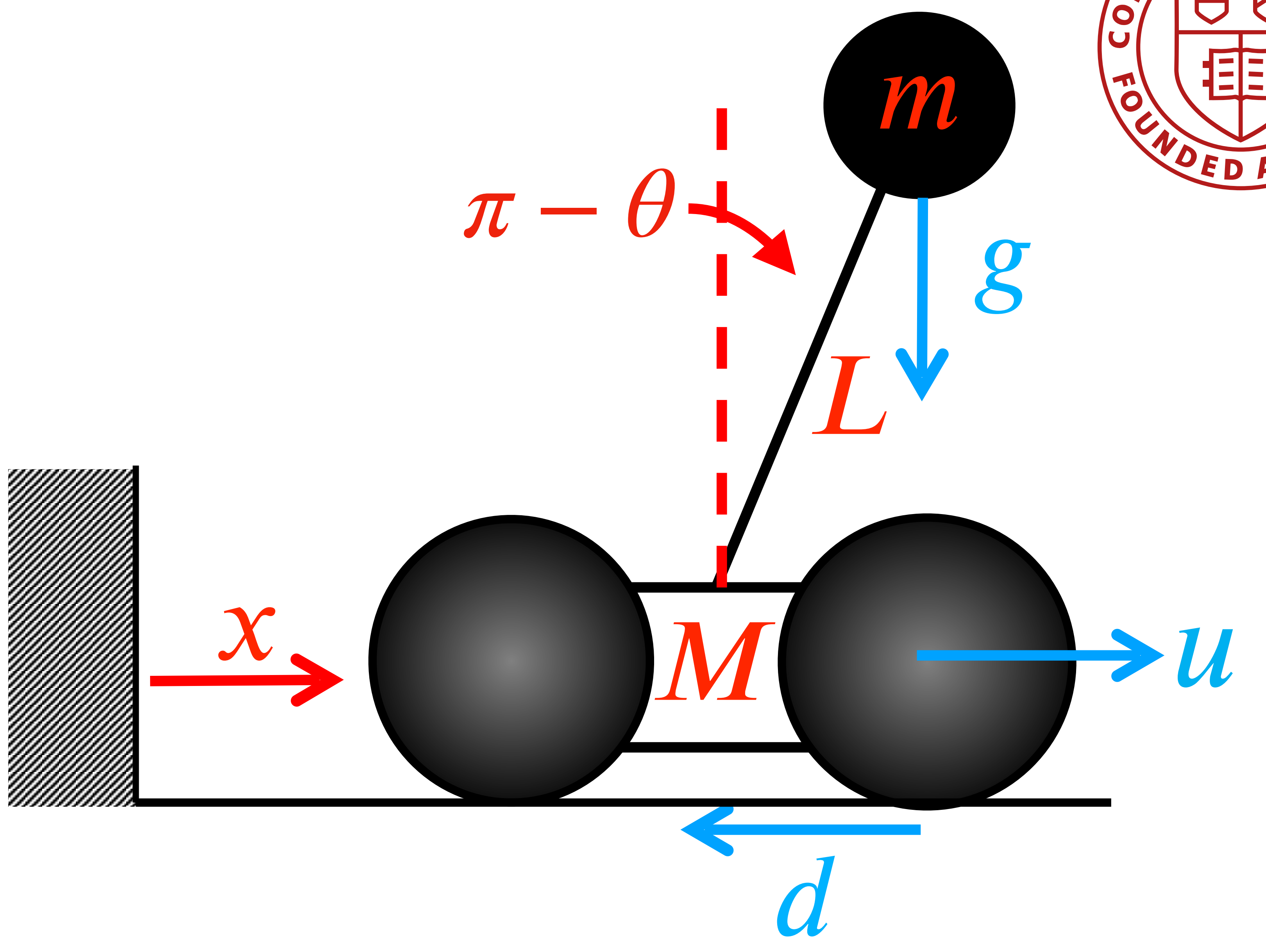
https://docs.scipy.org/doc/scipy/reference/generated/scipy.signal.place_poles.html

Linear Quadratic Regulator

- What are the optimal eigenvalues for our system?
  - Tradeoff performance and control effort
- Define cost function: $\int_0^\infty (x^T Q x + u^T R u) dt$
  - $Q = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 10 & \\ & & & 100 \end{bmatrix}$ cost of my state being away from setpoint
  - $R = 0.001$ cost of input energy
  - Solved using the Ricatti Equation (compute expensive $O(n^3)$)
- Matlab `>>lqr(A,B,Q,R)`

Inverted pendulum on a cart

The controller works!



Caveats:

- In simulation
- Pratical issues:
  - Imperfect models
  - Nonlinear parts: deadband, saturation, etc.
  - Partial state feedback

Review

- Linear system: $\dot{x} = Ax$
- Solution: $x(t) = e^{At}x(0)$
- Eigenvectors: $T = \begin{bmatrix} \xi_1 & \xi_2 & \dots & \xi_n \end{bmatrix}$
- Eigenvalues: $D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$

```
>>[T,D] = eig(A)
```

- Linear Transform: $AT = TD$
- Solution: $e^{At} = e^{TDT^{-1}t}$
- Mapping from x to z to x: $x(t) = Te^{Dt}T^{-1}x(0)$
- Stability in continuous time: $\lambda = a + ib$, stable iff $a < 0$
- Discrete time: $x(k+1) = \tilde{A}x(k)$, where $\tilde{A} = e^{A\Delta t}$
- Stability in discrete time: $\tilde{\lambda}^n = R^n e^{in\theta}$, stable iff $R < 1$
- Nonlinear systems: $\dot{x} = f(x)$
- Linearization: $\left.\frac{Df}{Dx}\right|_{\bar{x}}$
- Controllability: $\dot{x} = (A - BK)x$ `>>rank(ctrb(A,B))`
- Reachability
- Controllability Gramian
- Pole Placement `>>place(A,B,poles)`
- Optimal Control (LQR) `>>LQR(A,B,Q,R)`