Markov, Bayes Filter I

Fast Robots, ECE4160/5160, MAE 4190/5190

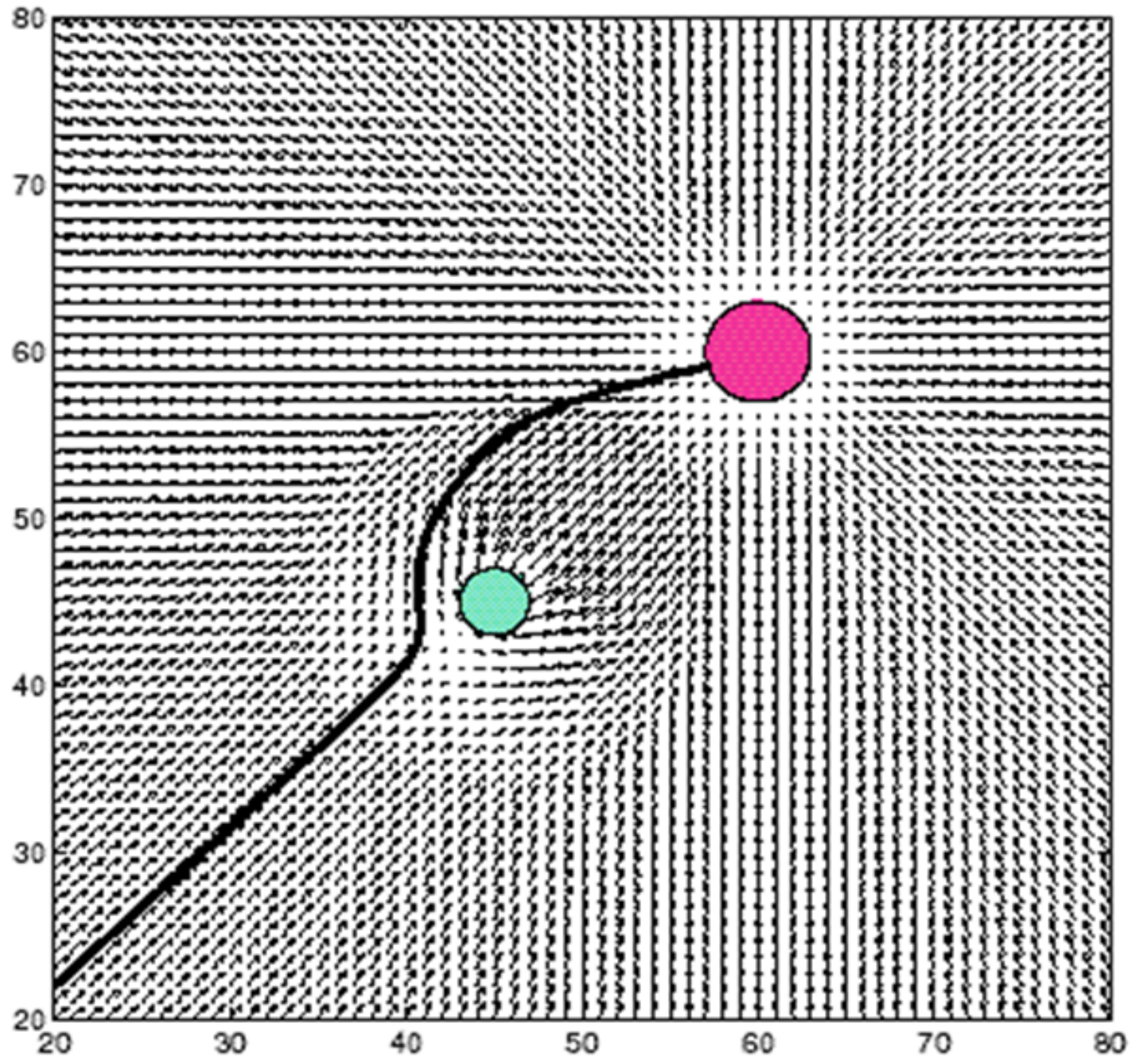
E. Farrell Helbling, 3/20/25

Class Action Items

- Lab 7: Kalman Filtering: please do not leave this to the weekend.
- Lab 8: Stunts. The lab is posted, you have two options, the flip or the drift. Please try to get this done next week before spring break (or do it after spring break). I don’t recommend taking the robot on a plane!
- Lab 3 regrade requests will close on **Thursday midnight.** Submit requests in canvas.
- Lab 4 grades delayed, found an error in the spreadsheet, will hopefully post grades later today!

Modeling path planning as a graph search problem

Real World → Configuration Space → Map Representation → Graph Construction → Graph Search



Common alternatives

- Optimal control
- Potential fields

Modeling path planning as a graph search problem

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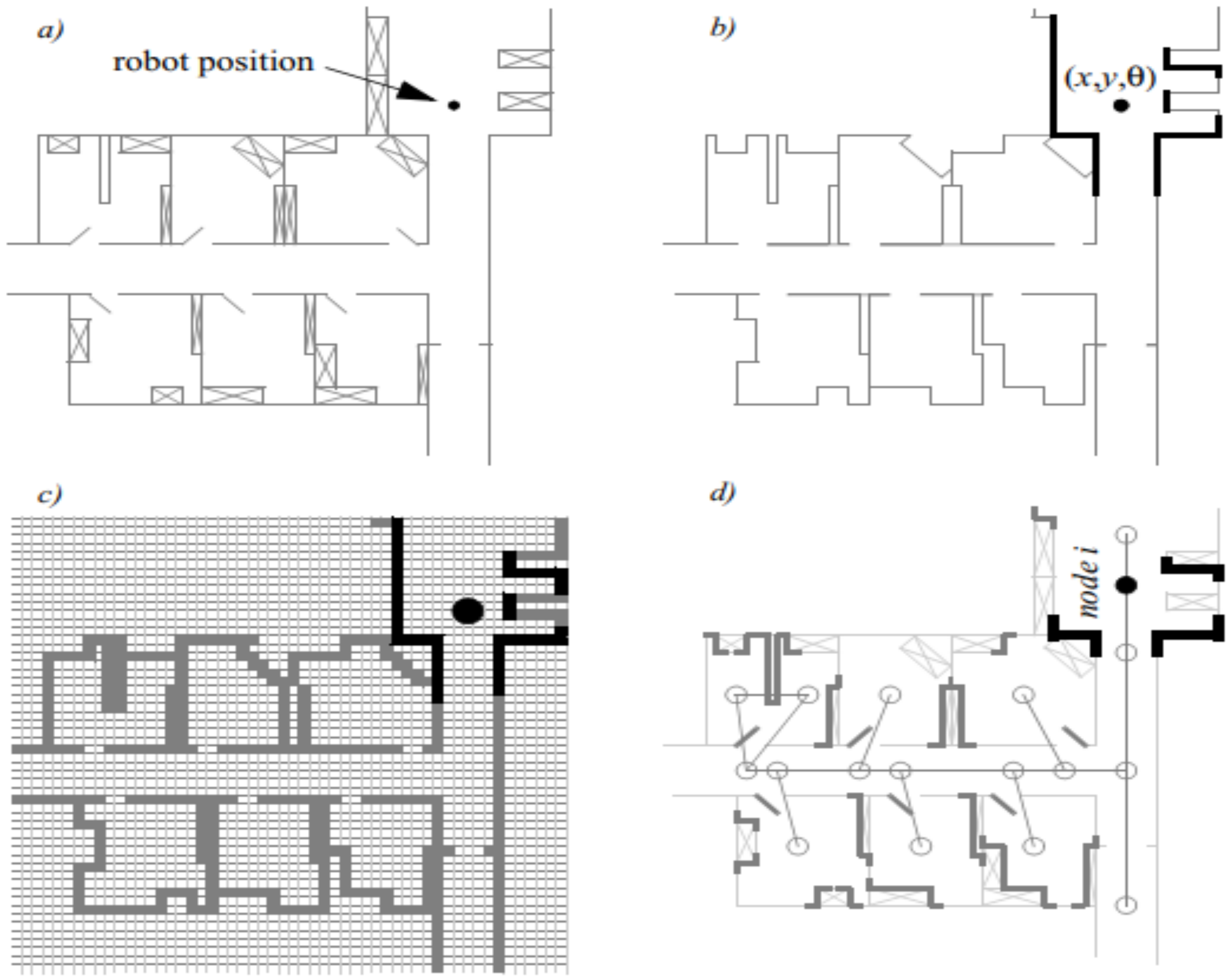
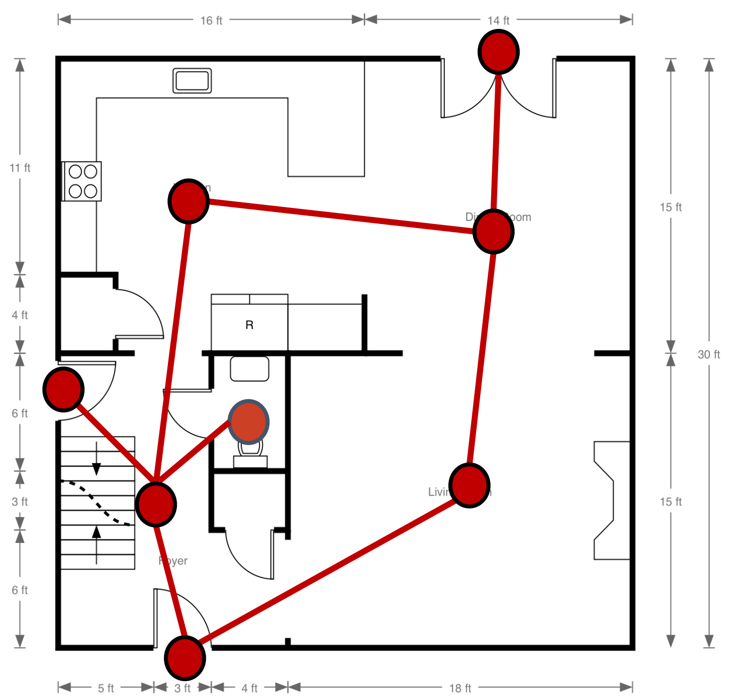
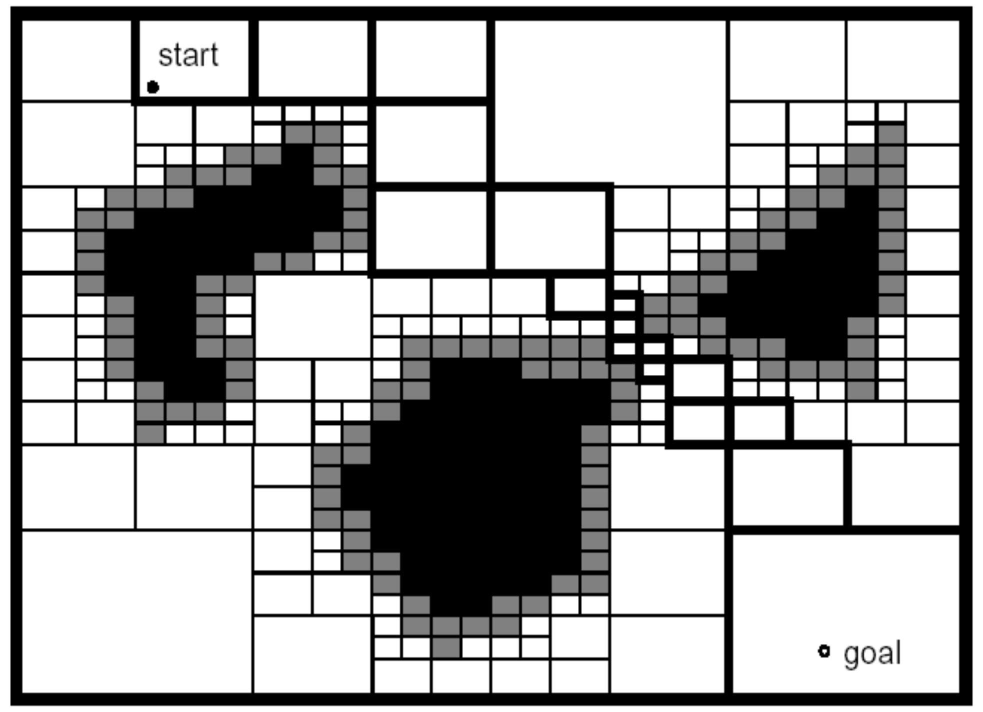
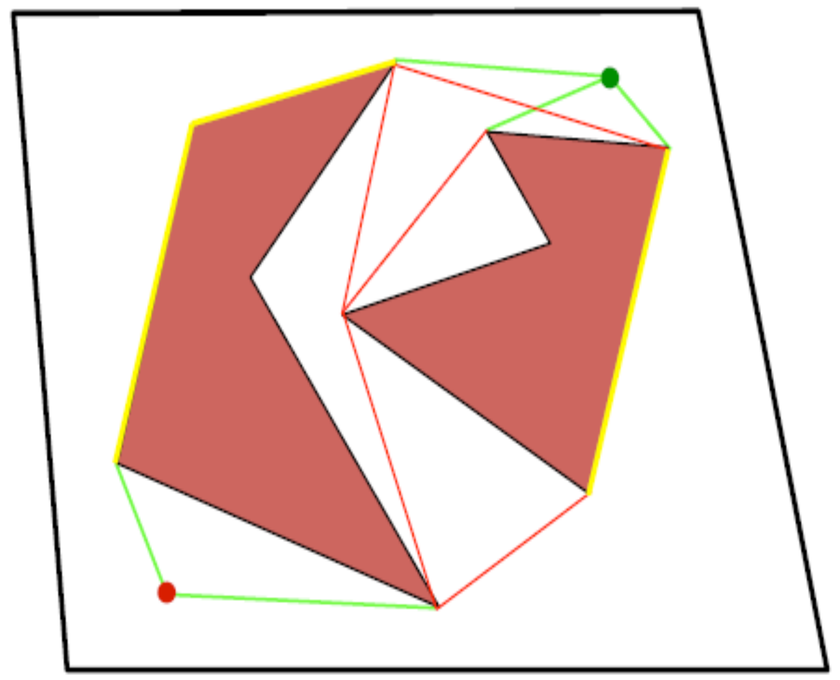


Modeling path planning as a graph search problem

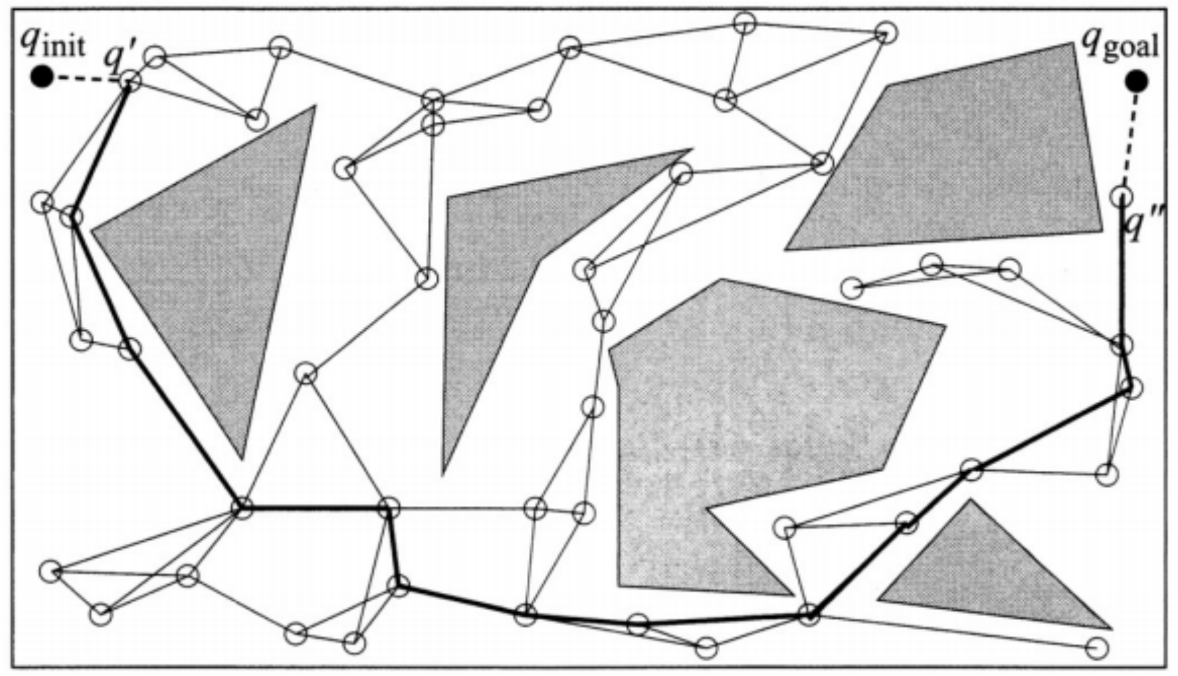
Real World → Configuration Space → Map Representation → **Graph Construction** → Graph Search

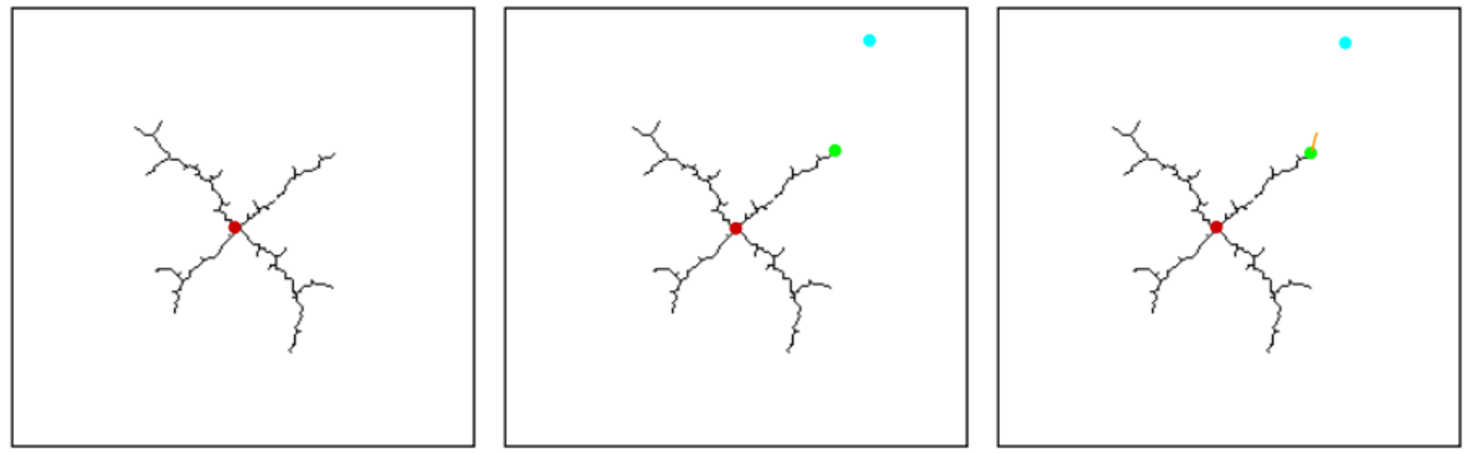




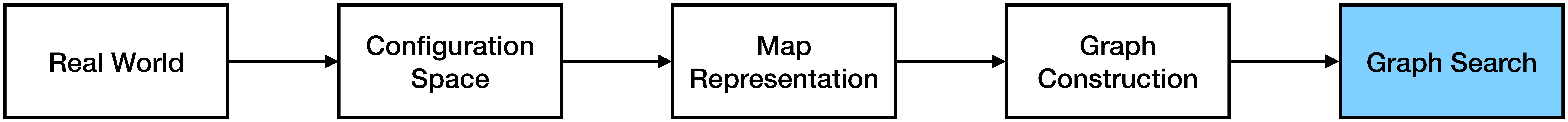


- Geometry-based graphs
  - Topological Graphs
  - Cell decomposition
  - Visibility Graphs
- Sampling-based graphs
  - RRT
  - PRM





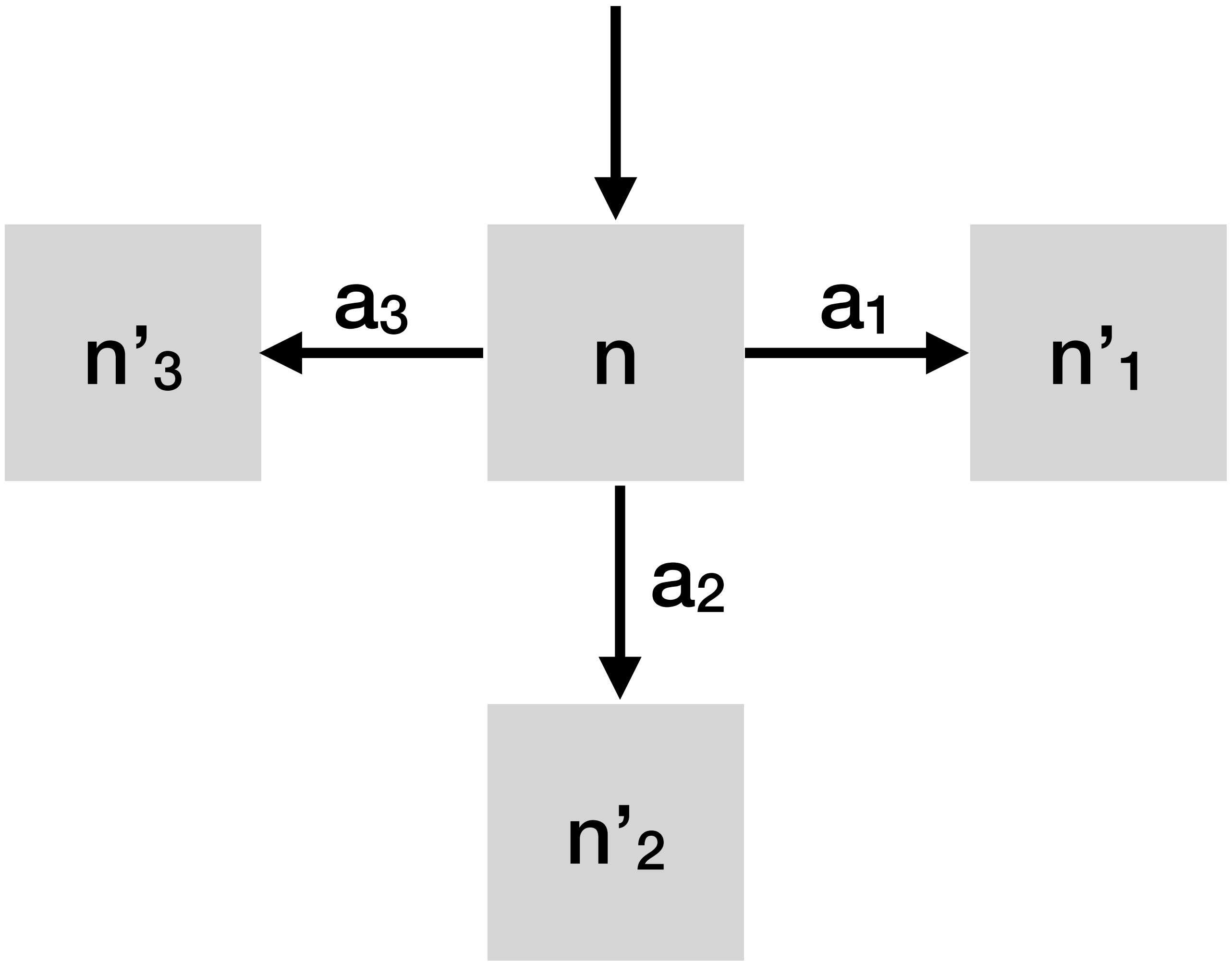
Modeling path planning as a graph search problem

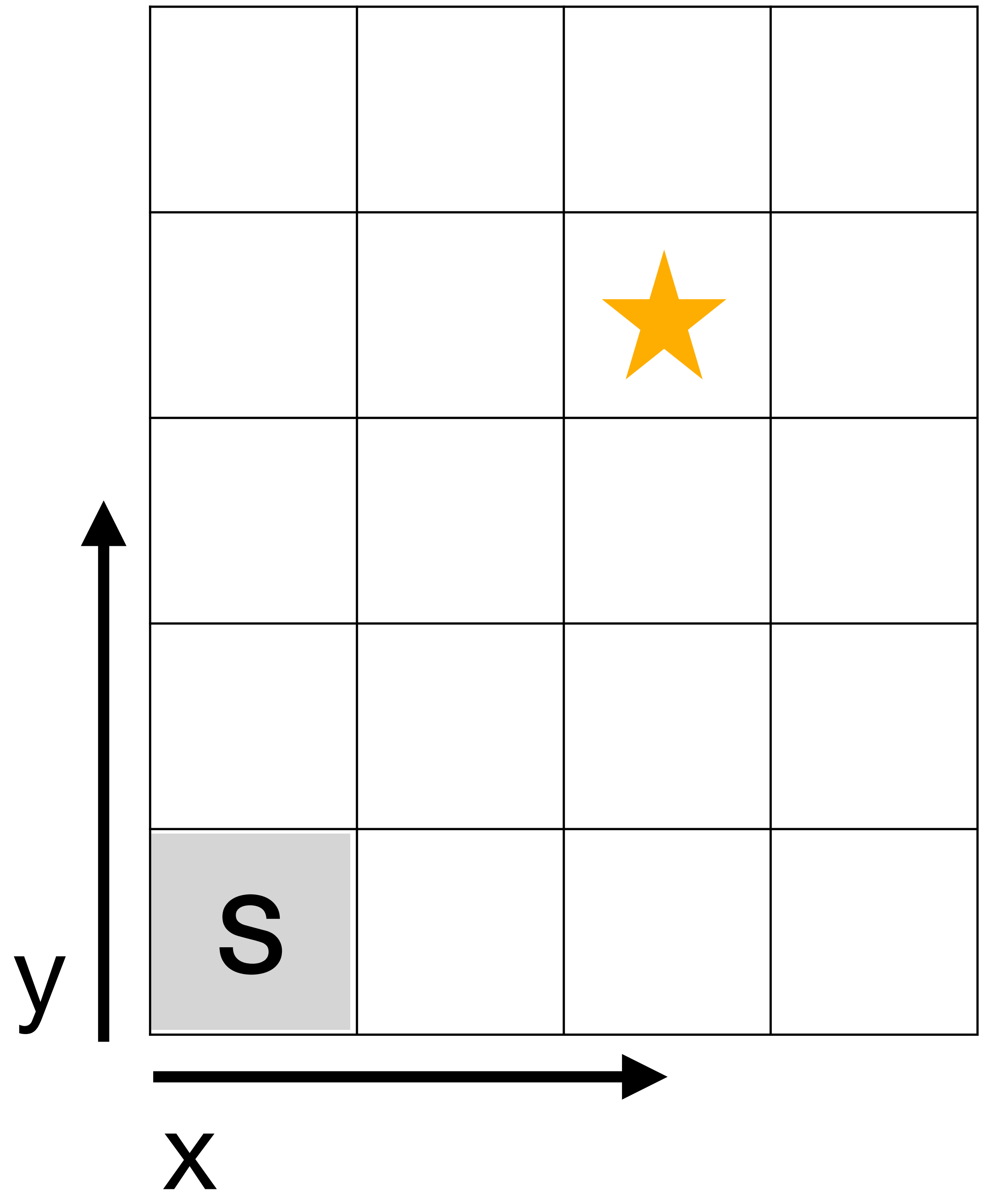


- Uninformed Searches
  - Breadth First
  - Depth First
  - Dijkstra’s (LCF)
- Informed Searches
  - Greedy
  - A*

Search Algorithms, General

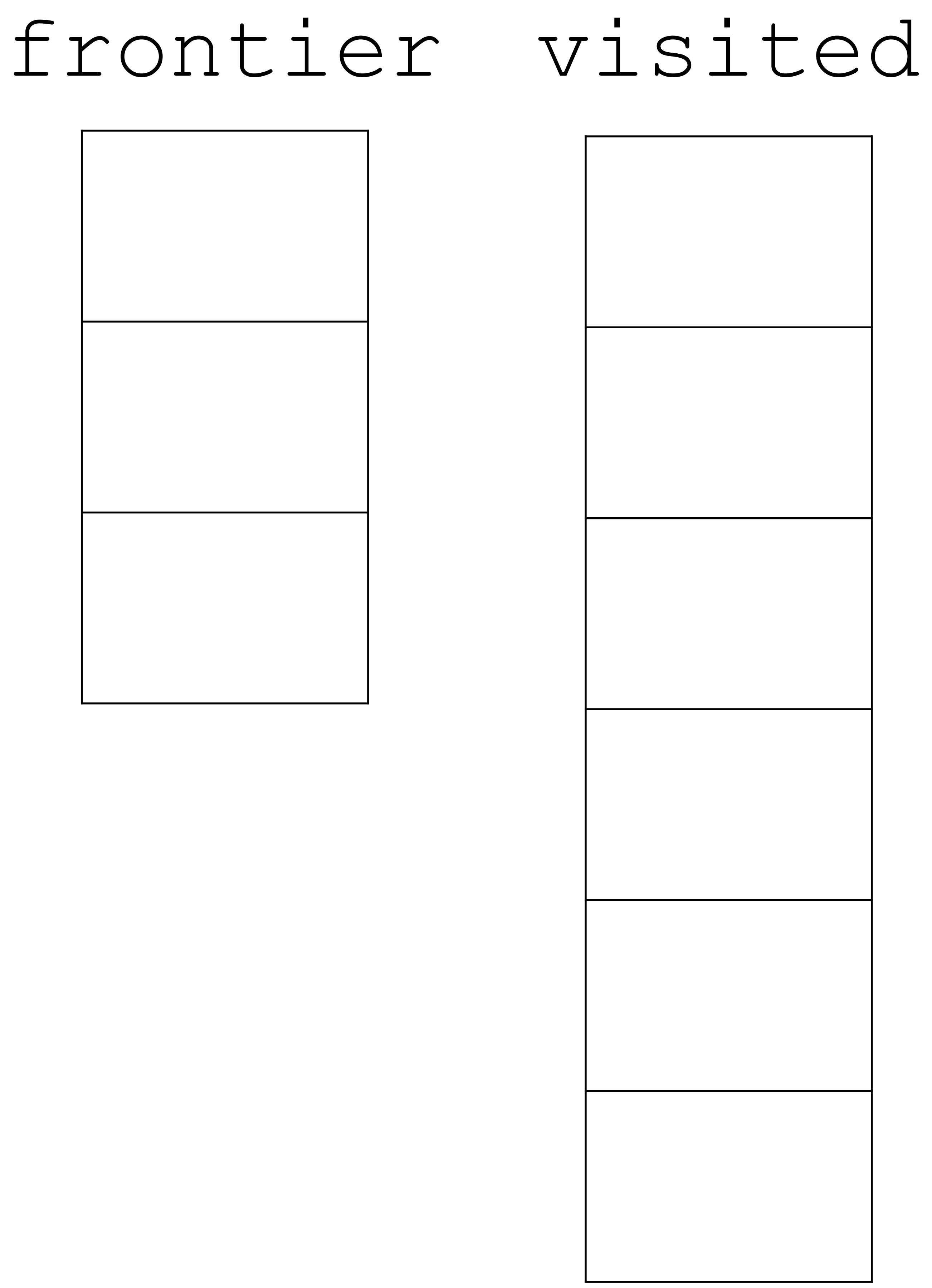
- For every node, n
- There is a set of actions, a
- That moves you to a new node, n’

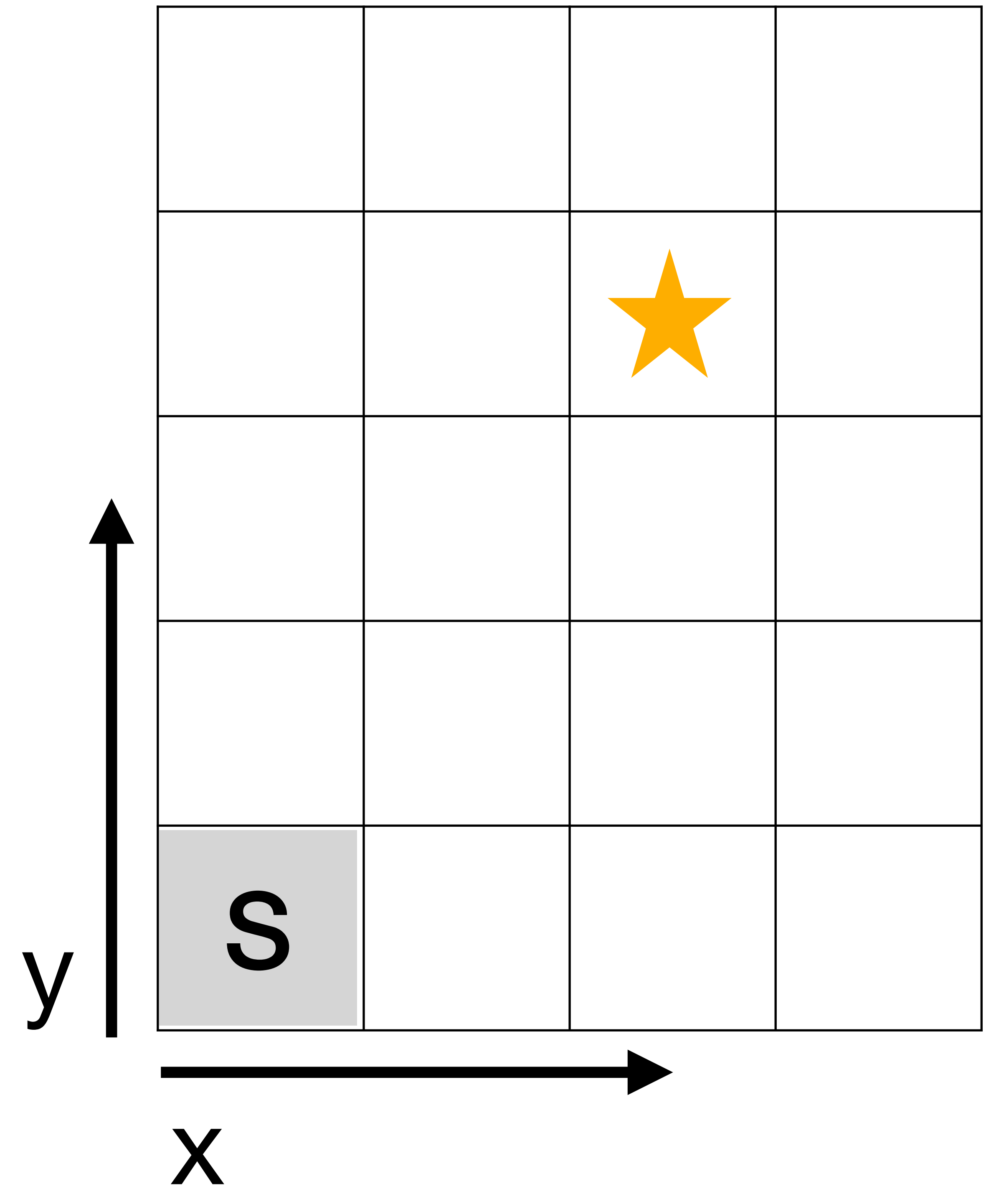




Uninformed Algorithms, General

```
n = state(init)
frontier.append(n)
while(frontier not empty)
  n = pull state from frontier
  append n to visited
  if n = goal, return solution
  for all actions in n
    n' = a(n)
    if n' not visited
     append n' to frontier
```





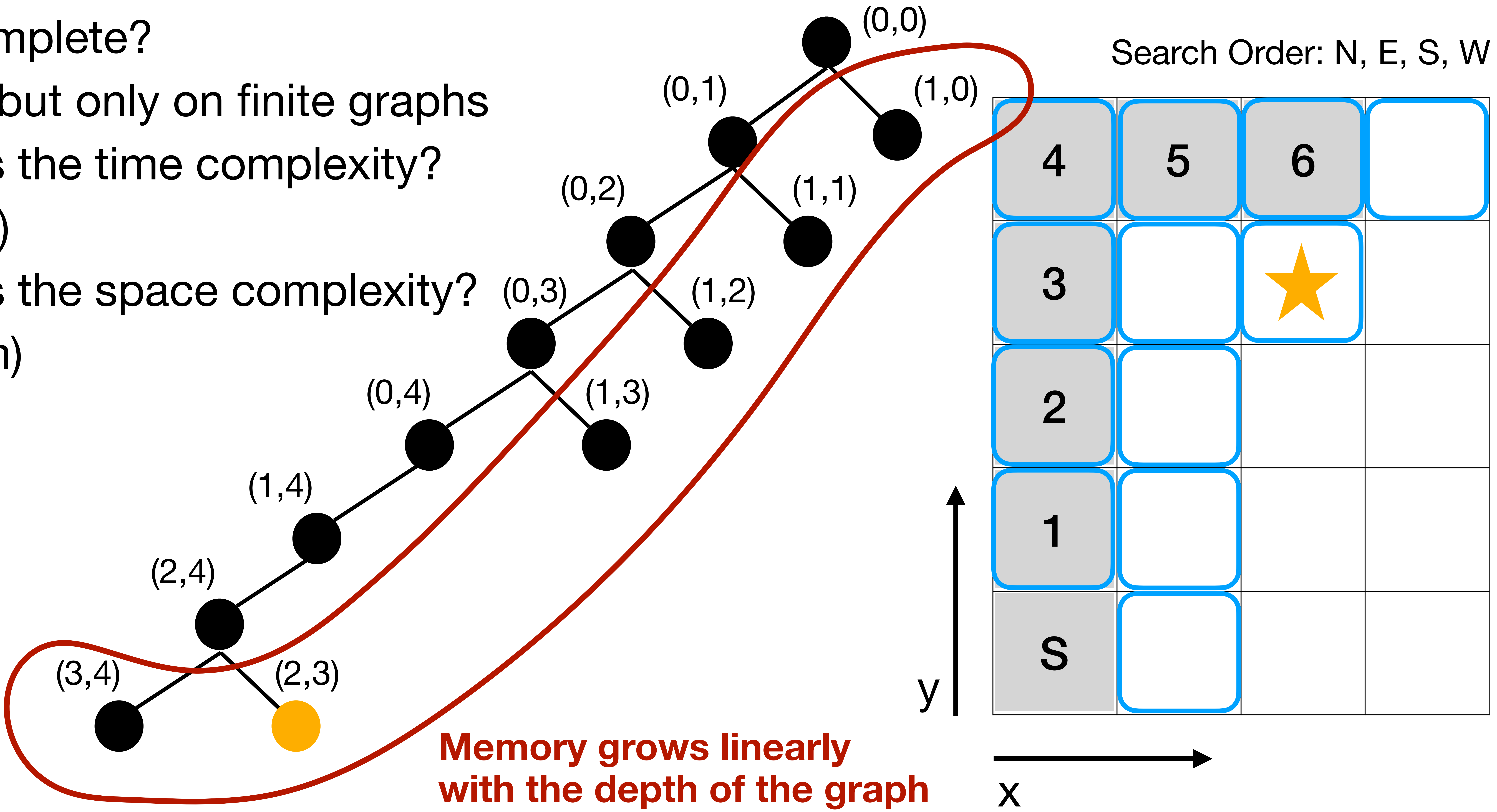
DFS: Last-In First-Out (LIFO)

BFS: First-In First-Out (FIFO)

LCFS: Prioritize cost

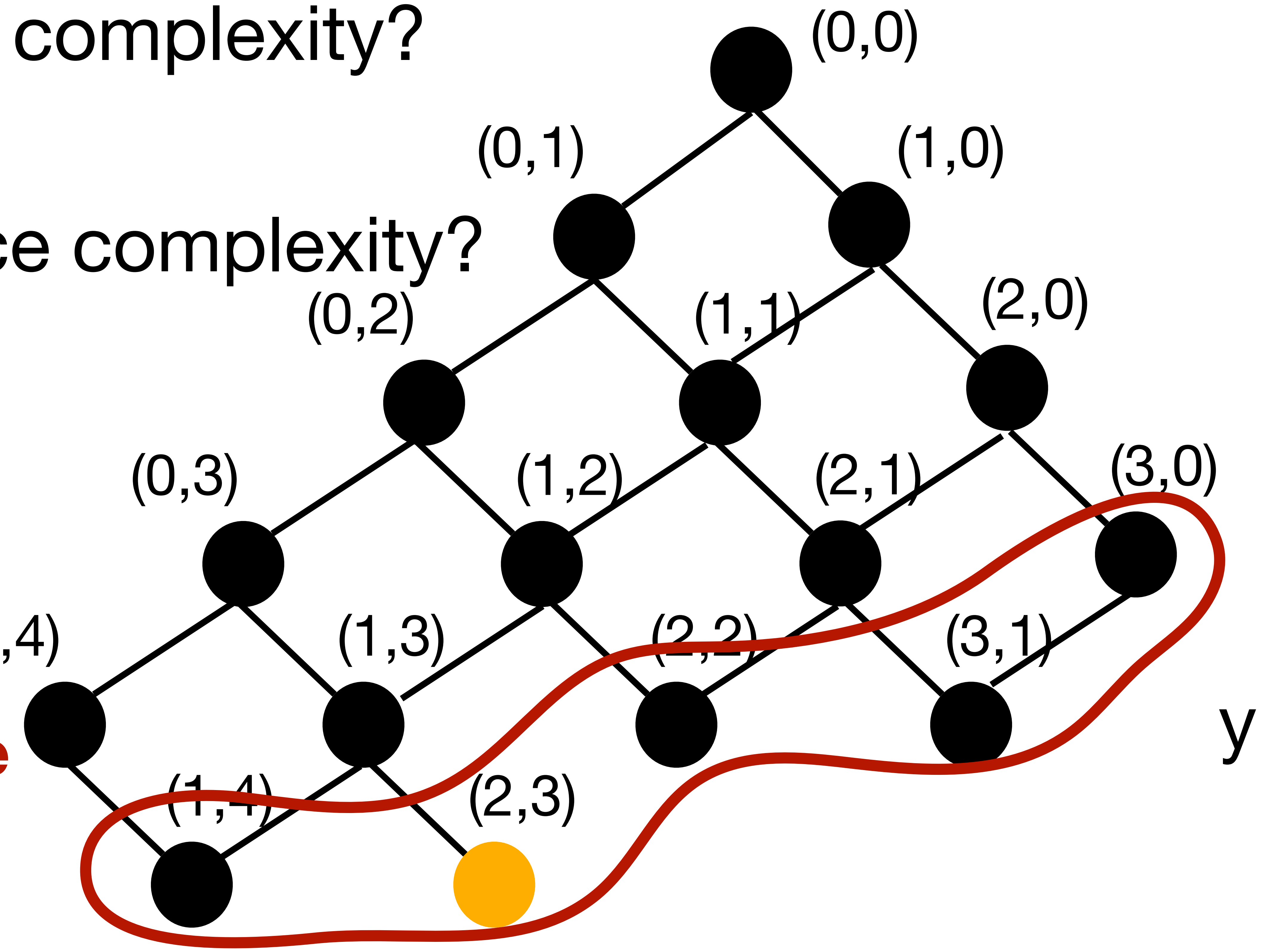
Depth First Search

- Is it complete?
  - Yes, but only on finite graphs
- What is the time complexity?
  - $O(b^m)$
- What is the space complexity?
  - $O(bm)$



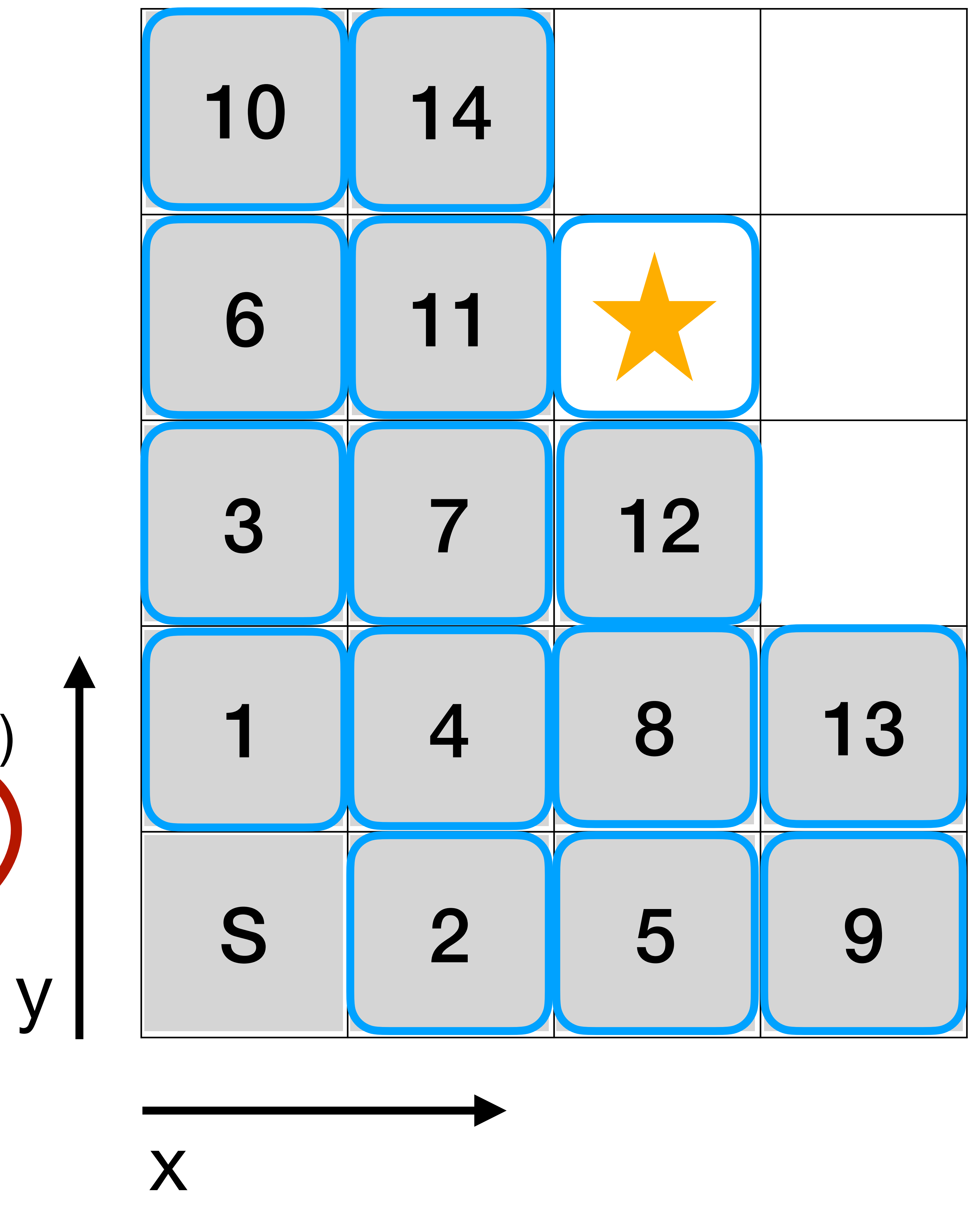
Breadth First Search

- Is it complete?
  - Yes, as long as b is finite
- Is it optimal?
  - Yes
- What is the time complexity?
  - $O(b^m)$
- What is the space complexity?
  - $O(b^m)$



Memory grows exponentially with the depth of the graph

Search Order: N, E, S, W

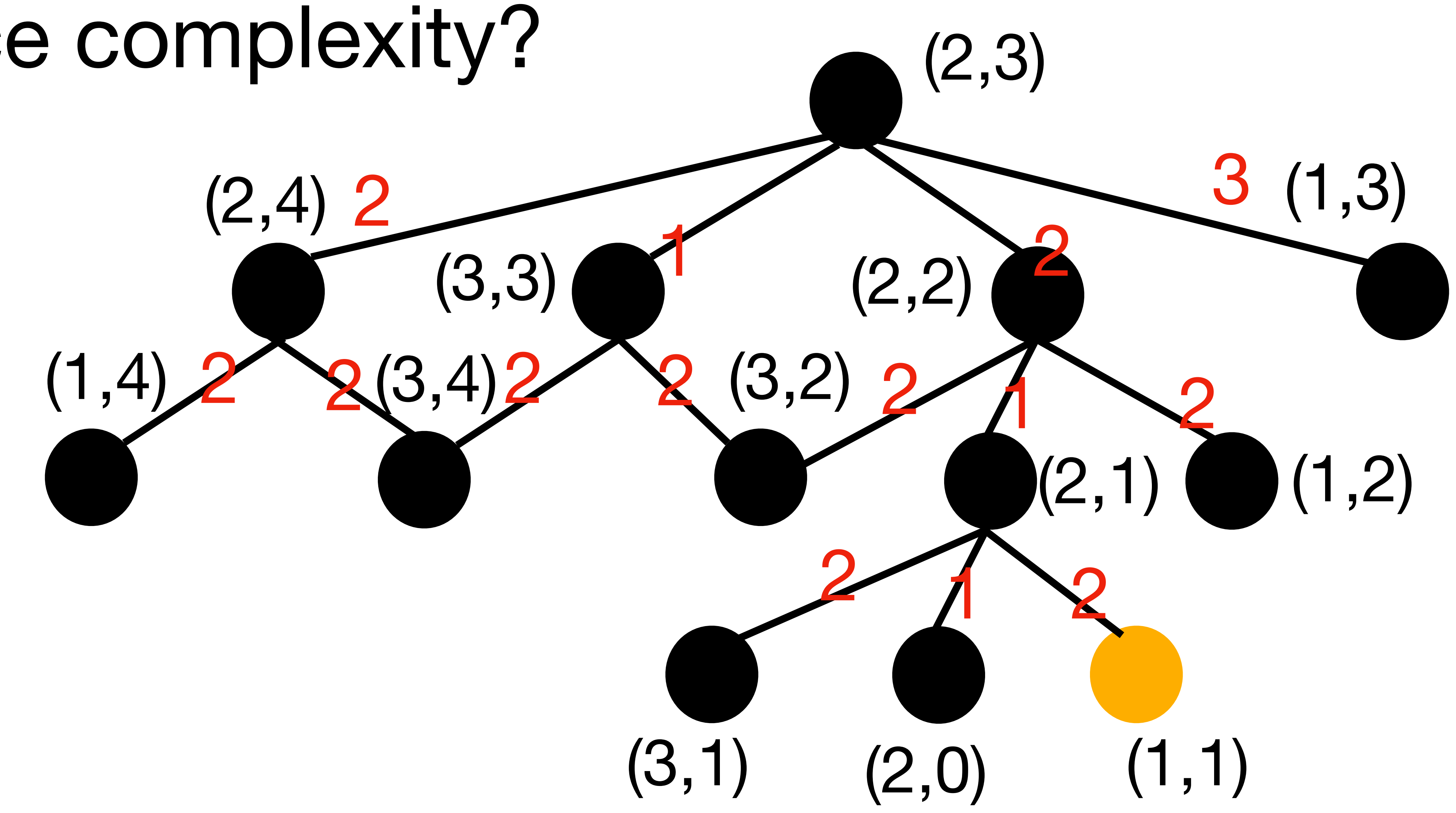


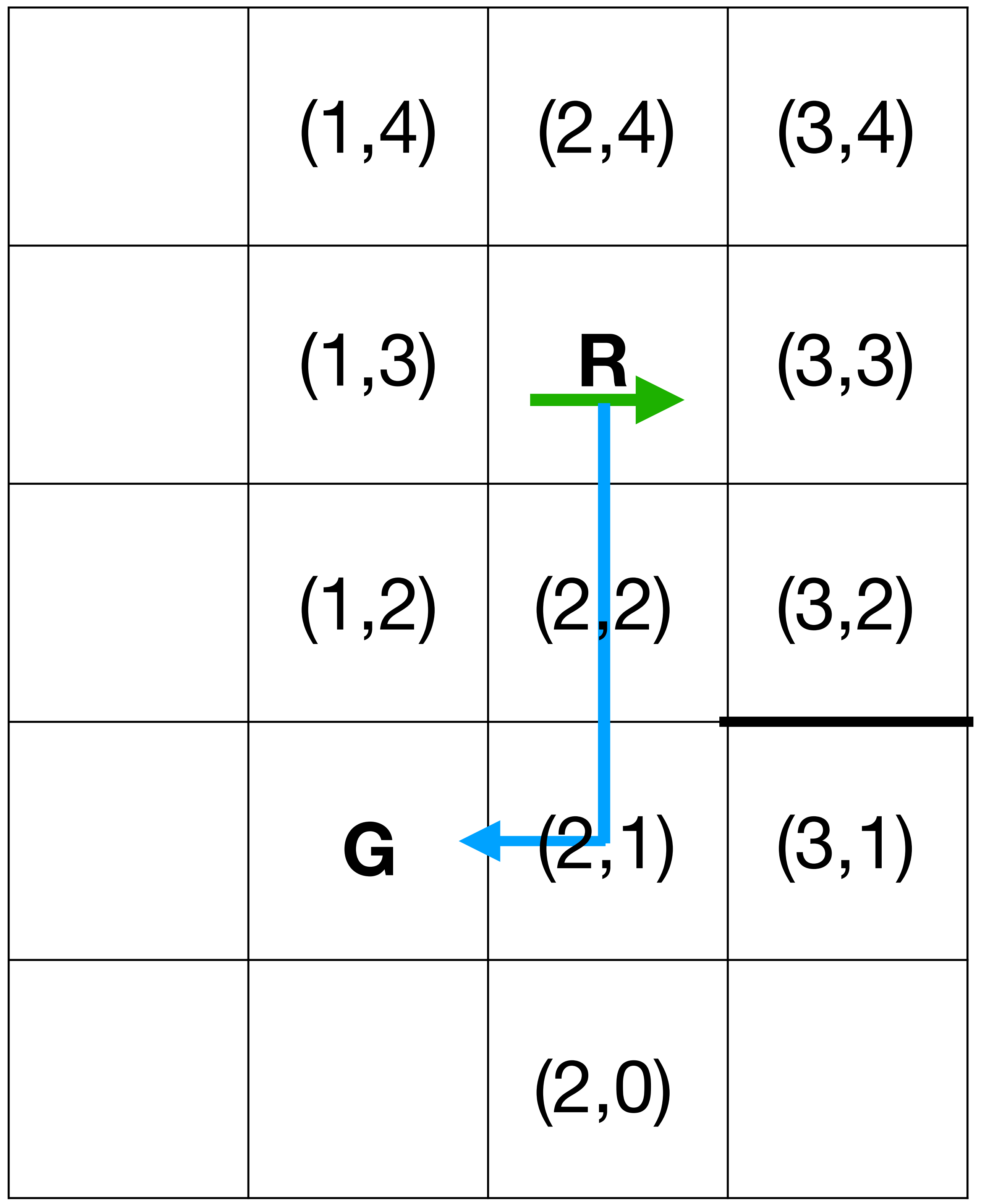
Lowest-Cost First Search (LCFS)

Consider parent cost!

- Is it complete?
  - Yes, as long as path costs are positive
- What is the time complexity?
  - $O(b^{1+C/c})$
- What is the space complexity?
  - $O(b^{1+C/c})$

- Go straight, cost 1
- Turn one quadrant, cost 1





Uninformed Search Algorithms

Criterion	BFS	DFS	LCFS
Complete	Yes (finite)	No (finite)	Yes (positive cost)
Time	$O(b^m)$	$O(b^m)$	$O(b^{1+C/c})$
Space	$O(b^m)$	$O(bm)$	$O(b^{1+C/c})$
Optimal	Yes (identical cost)	No	Yes
When to use	• Memory is a nonissue • Shallow solutions • Minimal branching factors • Shortest path needed	• Memory is restricted • Deep solutions	• Care about cost over length of path

Informed Search

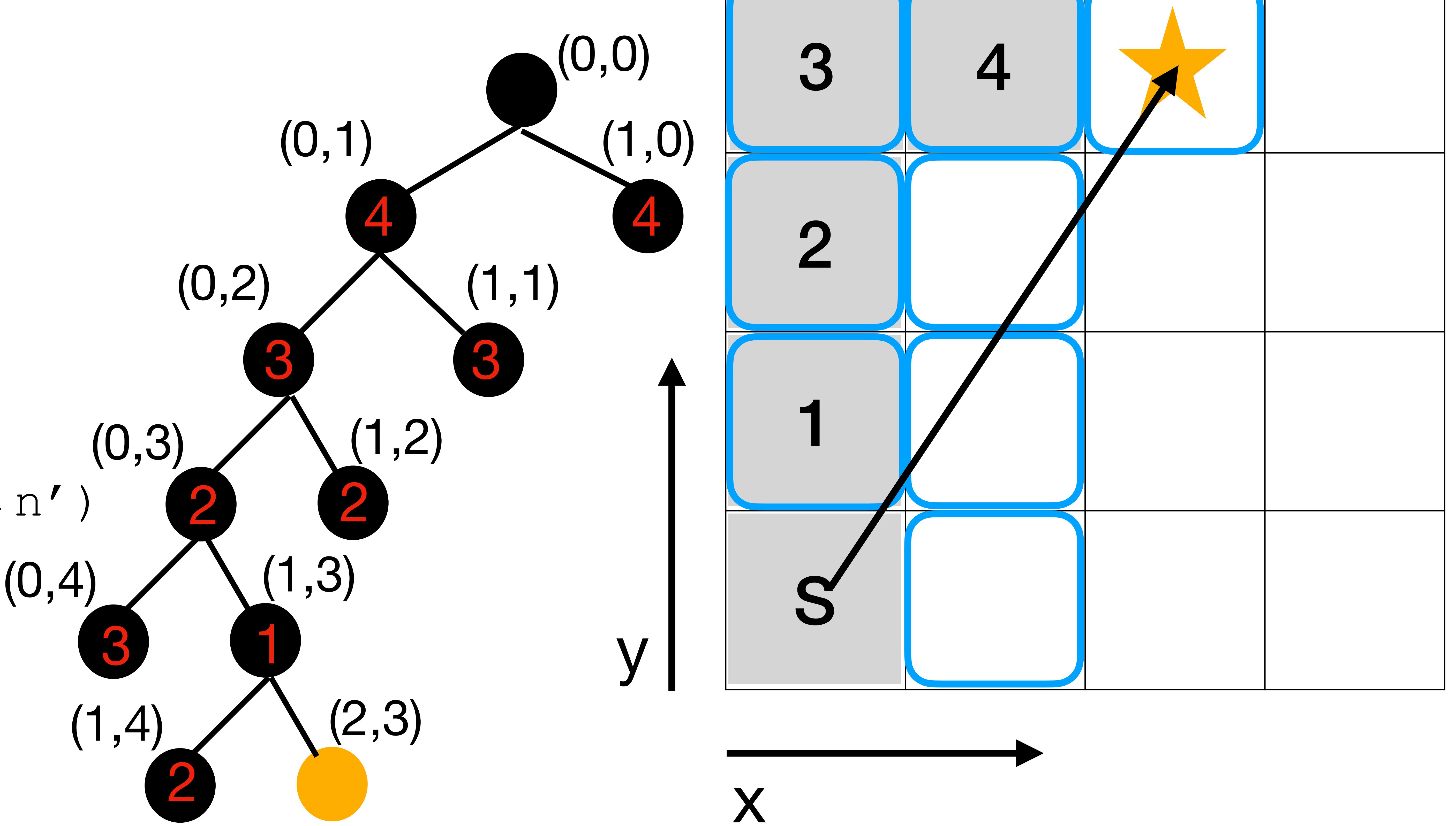
Greedy Search

```
n = state(init)
frontier.append(n)
while(frontier not empty)
  n = pull state from frontier
  visited.append(n)
  if n = goal, return solution
  for all actions in n
    n' = a(n)
    if n' not visited
        priority = heuristic(goal,n')
        frontier.append(priority)
```

Define a heuristic to target:

- Manhattan Distance
- $abs(x_s-x_g)+abs(y_s-y_g)$

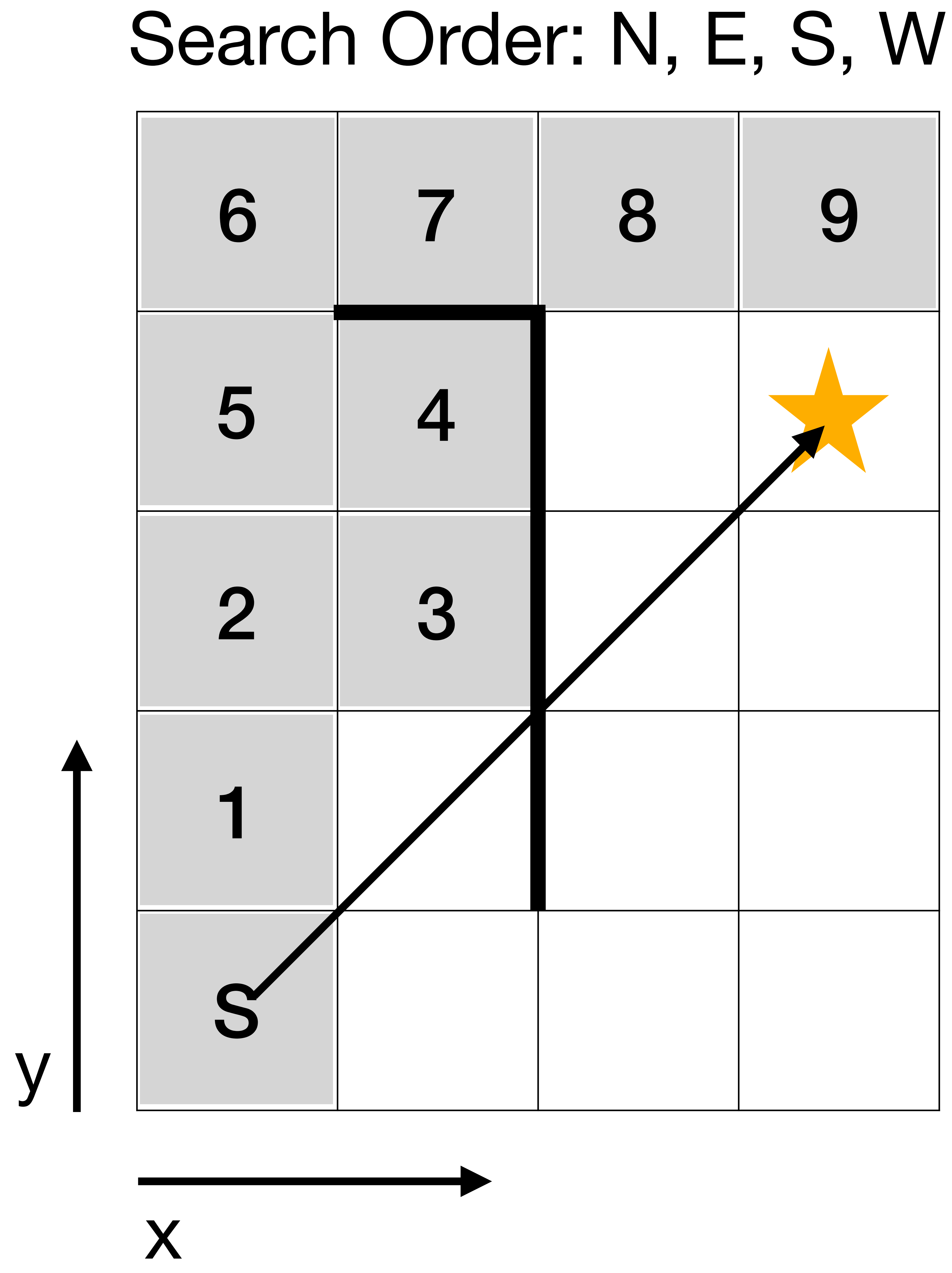
Search Order: N, E, S, W



Informed Search

Greedy Search

- Is it complete?
  - No
- What is the time complexity?
  - $O(b^m)$
- What is the space complexity?
  - $O(b^m)$
- Optimal?
  - No…



Search Algorithms, general

Can we do better? A*

- Breadth First Search
  - Complete and optimal
  - …but searches everything
- Lowest-Cost First Algorithm
  - Complete and optimal
  - … but it wastes time exploring in directions that aren't promising

Considers parent cost

- Greedy Search
  - Complete (in most cases)
  - … only explores promising directions

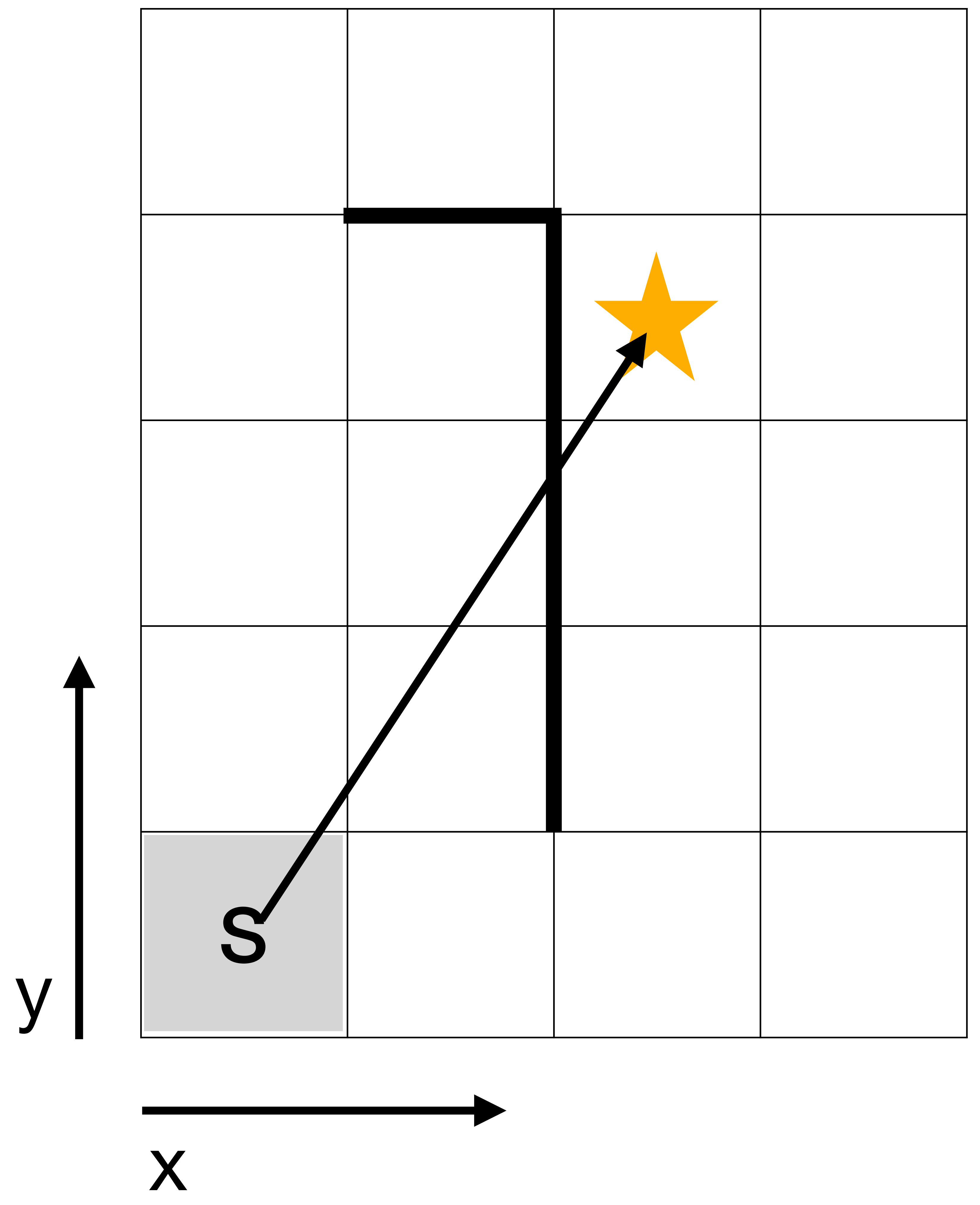
Considers goal

Informed Search

A* (A-star)

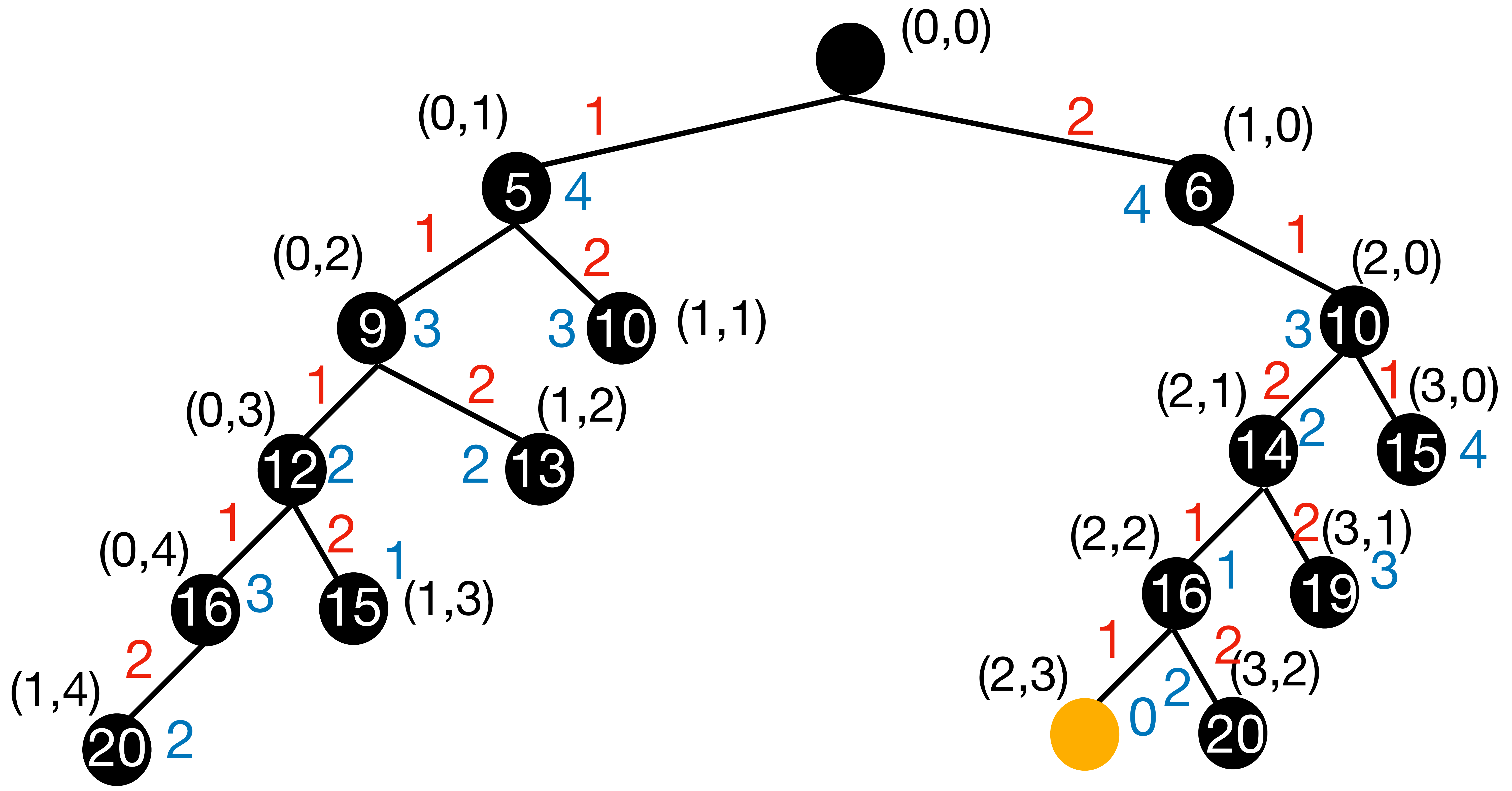
```
n = state(init)
frontier.append(n)
while(frontier not empty)
  n = pull state from frontier
  if n = goal, return solution
  for all actions in n
    n' = a(n)
    if (n' not visited)
       priority = heuristic(goal,n')+cost
       frontier.append(priority)
    if (visited and n'.cost < n_old.cost)
       visited.append(n')
```

Search Order: N, E, S, W

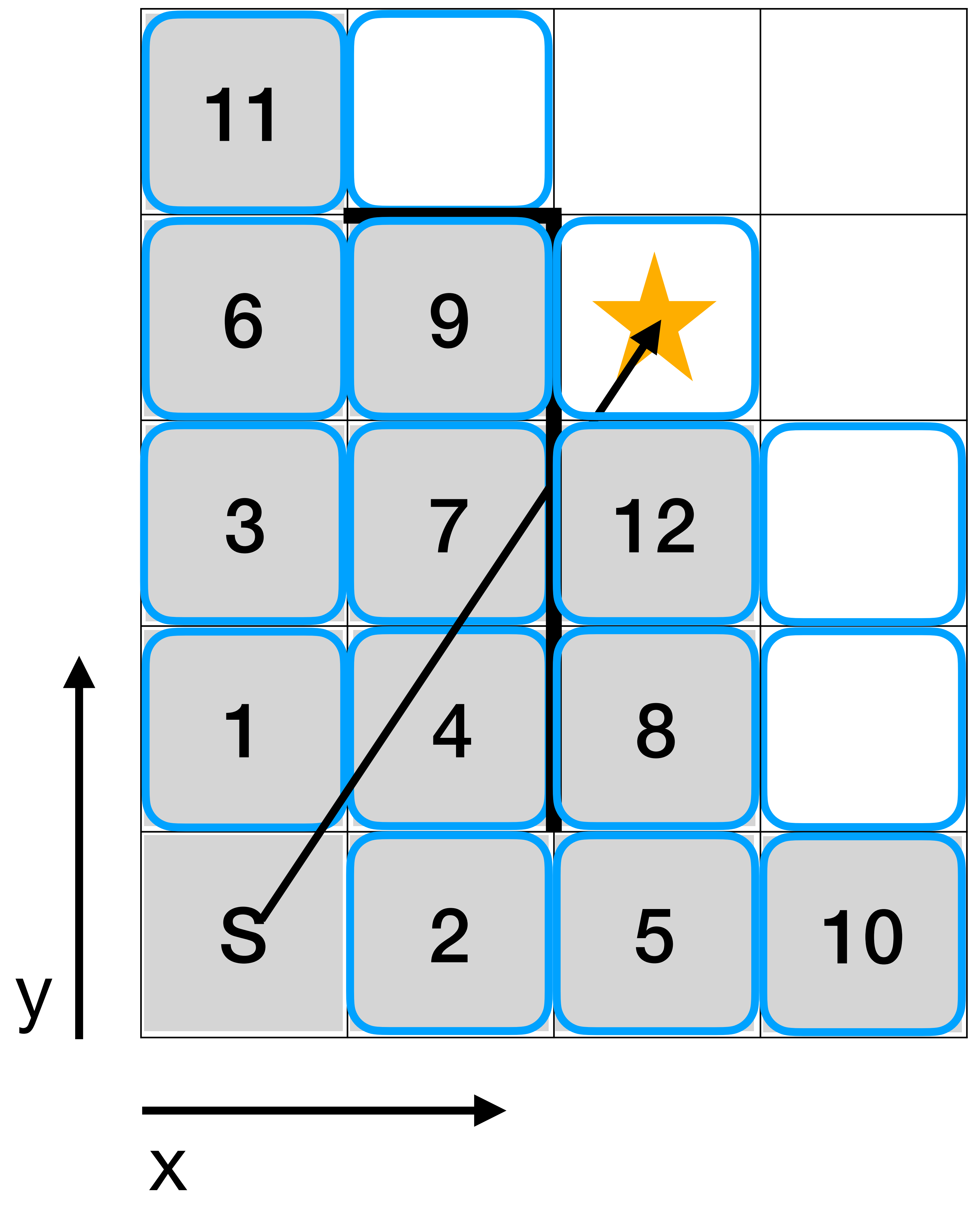


Informed Search

A* (A-star)



Search Order: N, E, S, W



A* Search

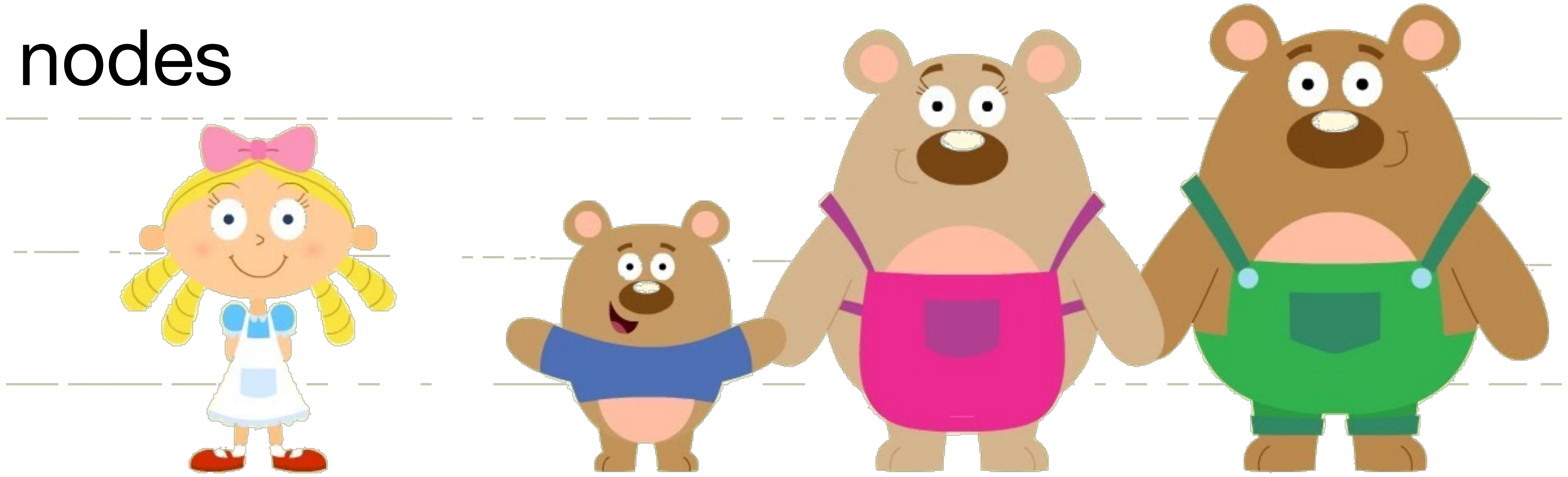
- What if the heuristic is too optimistic?
  - Estimated cost < true cost

Admissible heuristic

- What if the heuristic is too pessimistic?
  - Estimated cost > true cost
  - No longer guaranteed to be optimal

Inadmissible heuristic

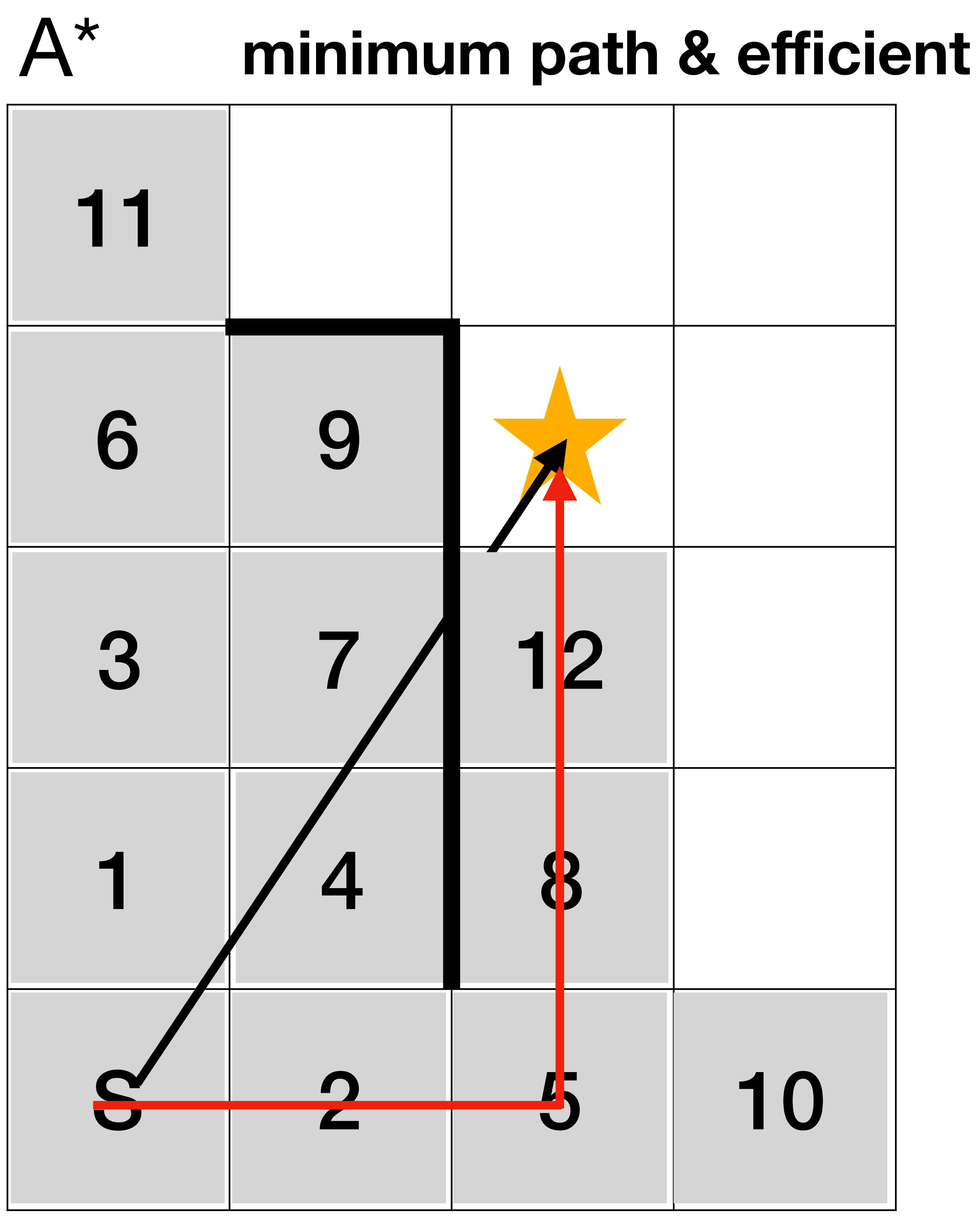
- What if the heuristic is just right?
  - Pre-compute the cost between all nodes
  - Feasible for you?



Informed Search

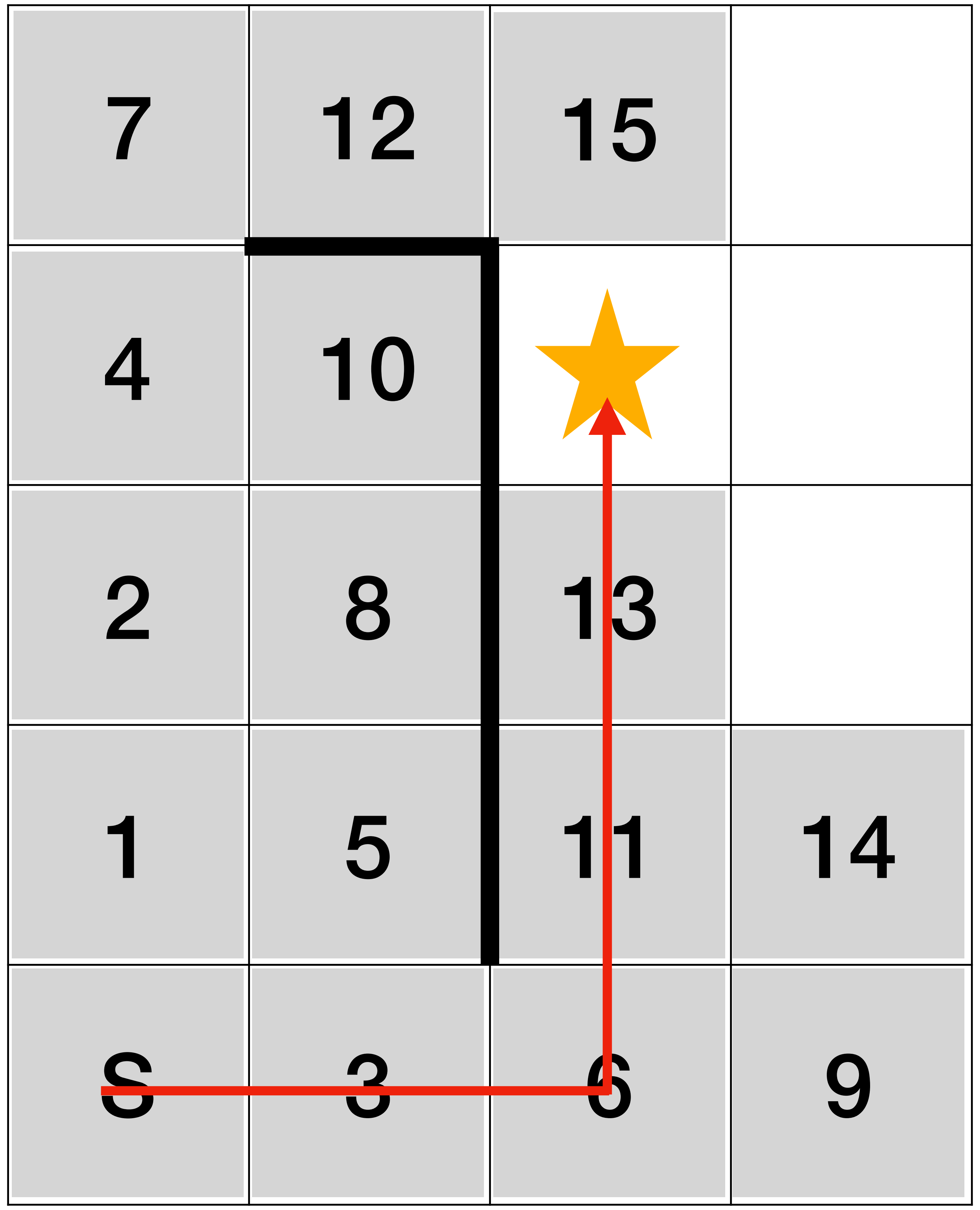
A* (A-star)

- Is it complete?
  - Yes!
- What is the time complexity?
  - $O(b^m)$
- What is the space complexity?
  - $O(b^m)$
- Optimal?
  - Yes, if the heuristic is admissible!

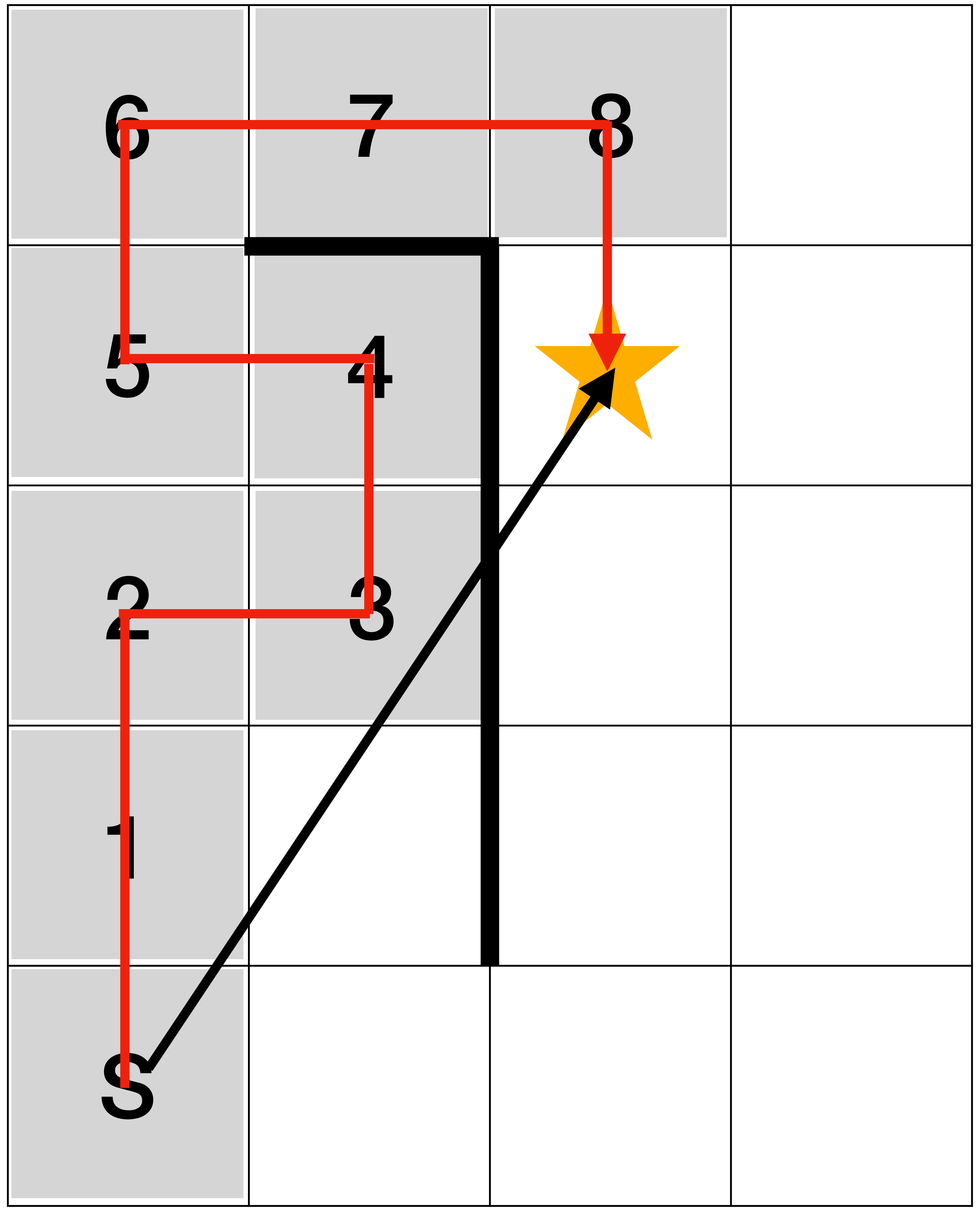


Summary

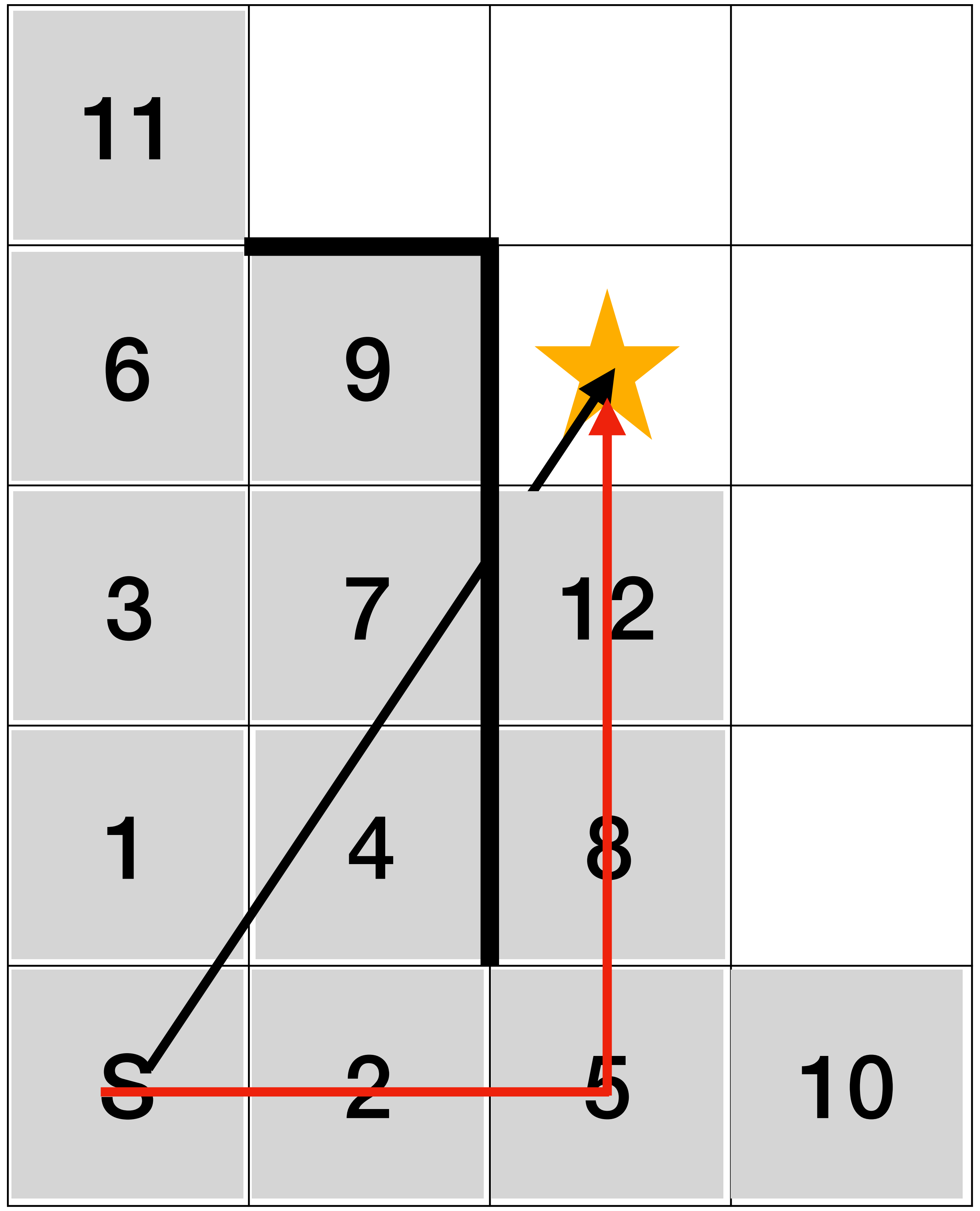
LCFS minimum path



Greedy



A* minimum path & efficient



Bayes Theorem

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Robot-Environment Model

+

Markov Assumption

=

Bayes Filter

Bayes Theorem

+

Robot-Environment Model

+

Markov Assumption

=

Bayes Filter

Bayesian Inference

Likelihood

Posterior

$$\boxed{P(x|z)} = \frac{\boxed{P(z|x)}\boxed{P(x)}}{\boxed{P(z)}}$$

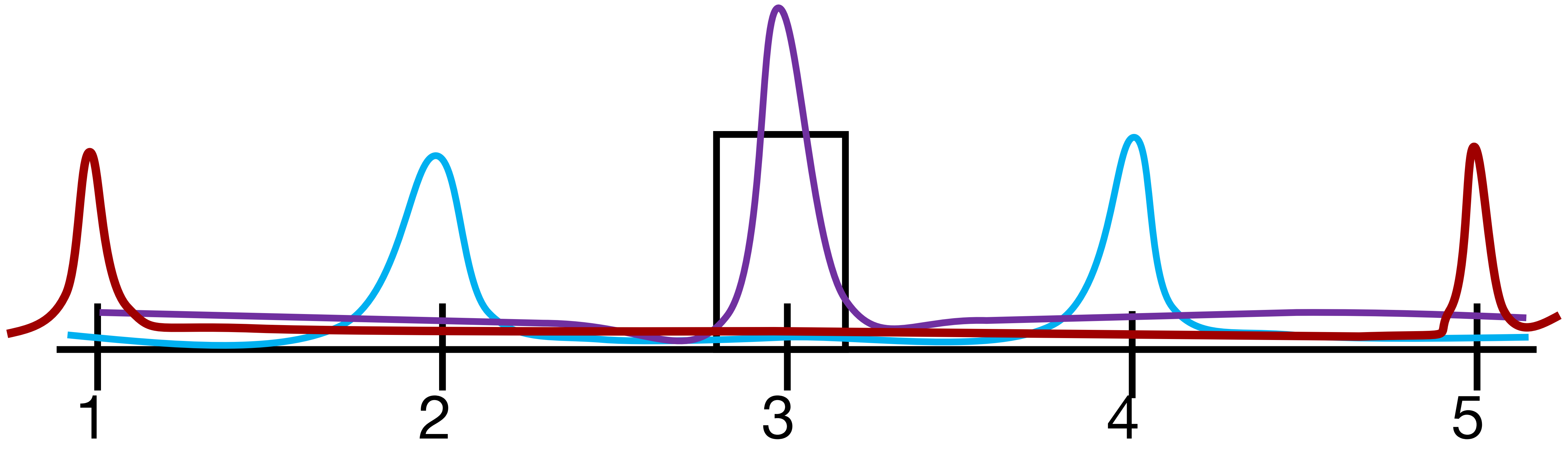
Prior

Marginal Likelihood (constant)

x: robot pose

z: sensor data

- Lost robot example
  - Sensor measures distance to the door
  - $p(X_0 = 1 \text{ or } 2 \text{ or } 3 \text{ or } 4 \text{ or } 5) = 1/5$
  - $p(x|z)$ can be hard to compute
  - What is $p(z|x)$?
  - If $Z = 1$, where are you most likely to be?
  - If $Z = 0$, where are you most likely to be?
  - If $Z = 2$, where are you most likely to be?



Bayes Theorem

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Robot-Environment Model

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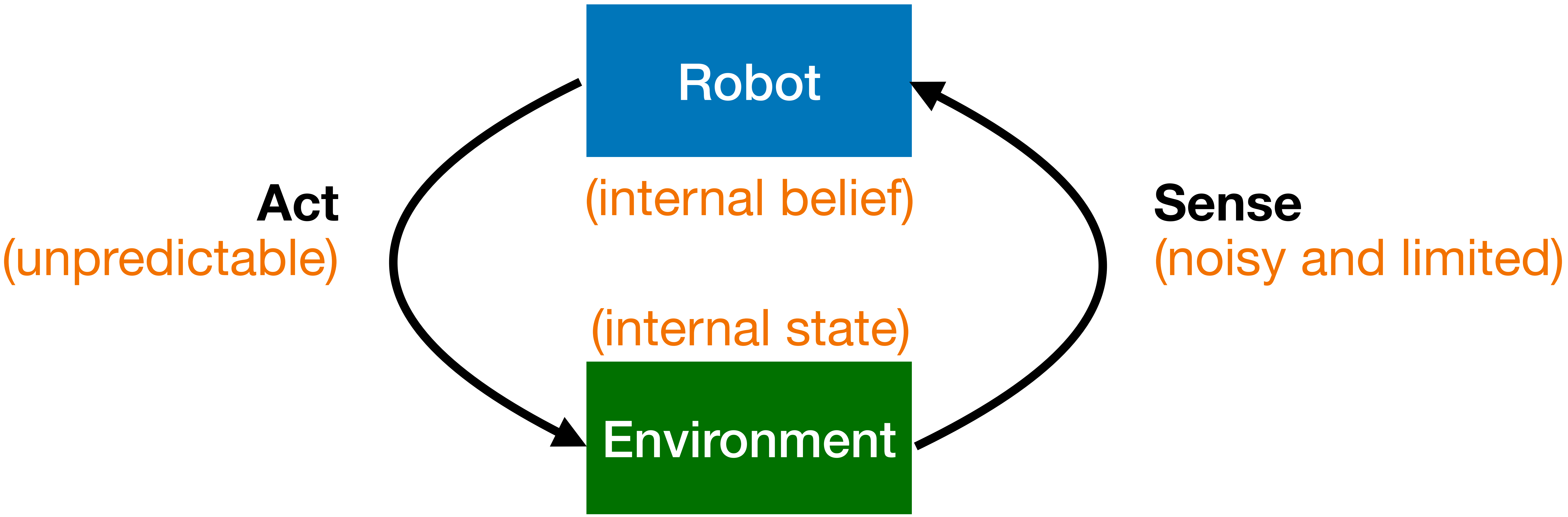
Markov Assumption

=

Bayes Filter

Robot-Environment Interaction



- Two fundamental types of interaction between a robot and its environment:
  - Sensor measurements/ observations
  - Control actions

Robot-Environment Model

- Helps us express a robot-environment interaction using probability
  - Typically modeled as a discrete time system
    - The **state** at time t will be denoted as x_t
    - A **sensor measurement** at time t will be denoted as z_t
    - A **control action** will be denoted as u_t
      - Induces a transition from x_{t-1} to x_t

Conventions as per Siegwart, R., Nourbakhsh, I.R. and Scaramuzza, D., 2011.
Introduction to autonomous mobile robots. MIT press.

Robot-Environment Model

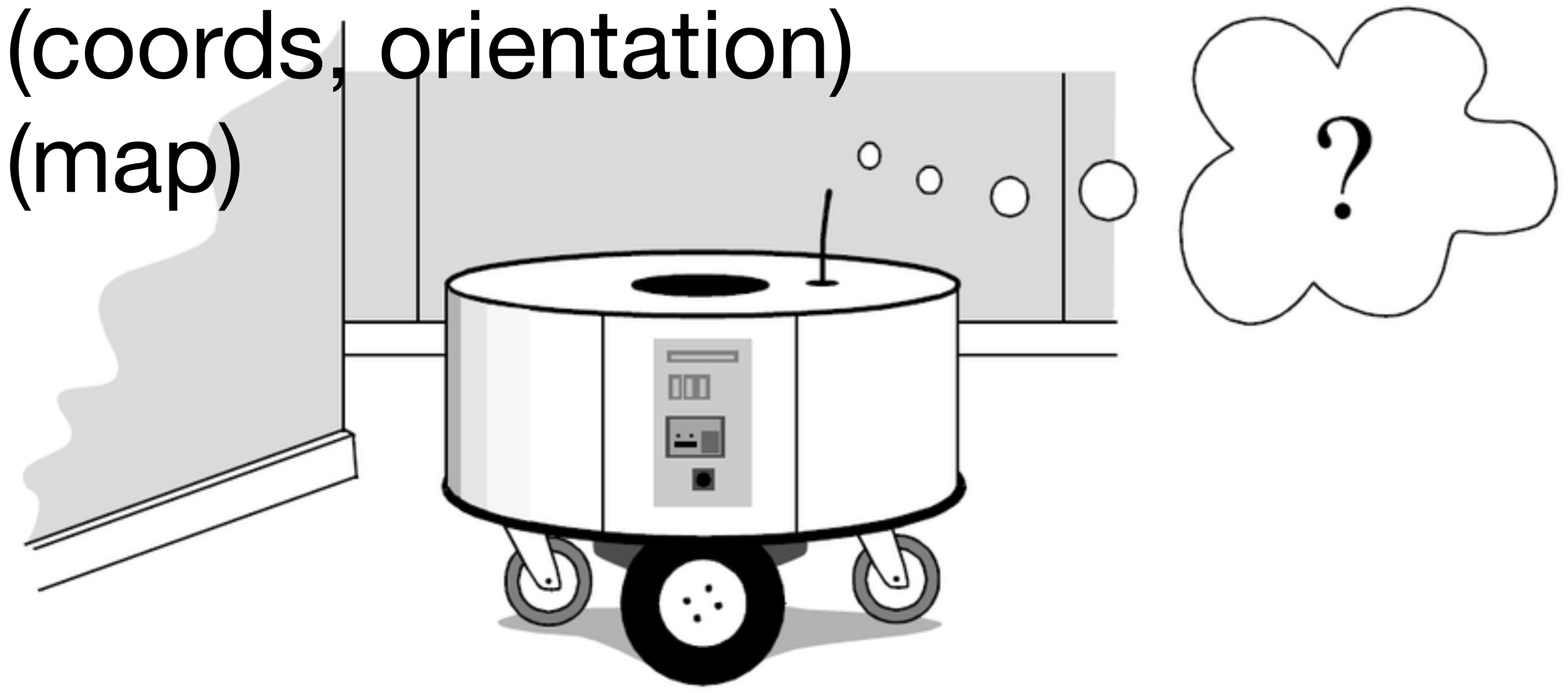
Assumptions (arbitrary)

- The robot executes a control action u_t first and then takes a measurement z_t
- There is one control action u per time step t
    - This includes the legal action “do-nothing”
- There is only one measurement z per time step t
- Shorthand notation: $x_{t1:t2} = x_{t1}, x_{t1+1}, x_{t1+2}, \ \dots, x_{t2}$

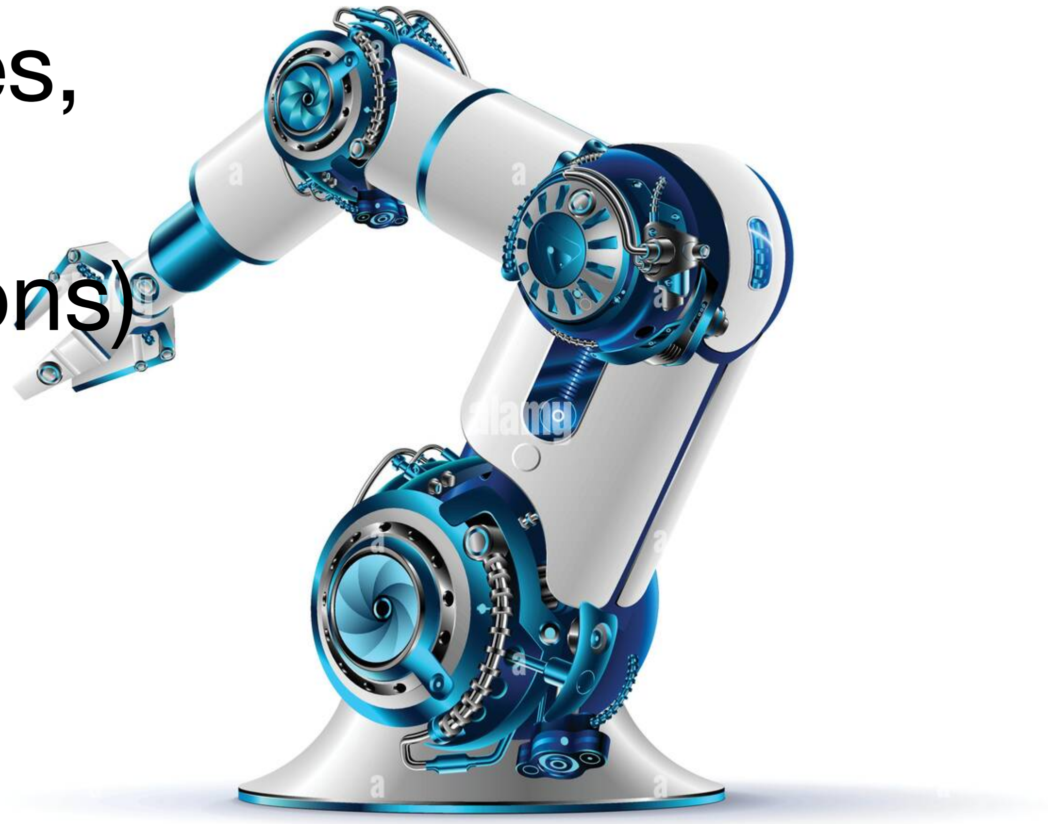
Robot State

- Robot-specific:
  - Pose, velocity, sensor status, etc.
- Environment-specific:
  - Static variables: locations of walls, etc.
  - Dynamic variables: people, etc.
- … context specific

(coords, orientation)
(map)



(joint angles,
velocities,
accelerations)
(objects,
texture)



Sensor Measurements/ Observations

- Tend to increase the robot's knowledge

balance

Control Actions

- … change the state of the world
- Carry information about the change of robot state in the time interval $(t-1:t]$
- Tends to induce loss of knowledge

Probabilistic Generative Laws

- The evolution of state and measurements is governed by probabilistic laws
  - State: How is x_t generated stochastically?
  - Measurements: How is z_t generated stochastically?
- State generation
  - x_t depends on $x_{0:t-1}$, $z_{1:t-1}$, and $u_{1:t}$
  - $p(x_t \mid x_{0:t-1}, z_{1:t-1}, u_{1:t})$ **...intractable!**

Bayes Theorem

+

Robot-Environment Model

+

Markov Assumption

=

Bayes Filter

Markov Assumption

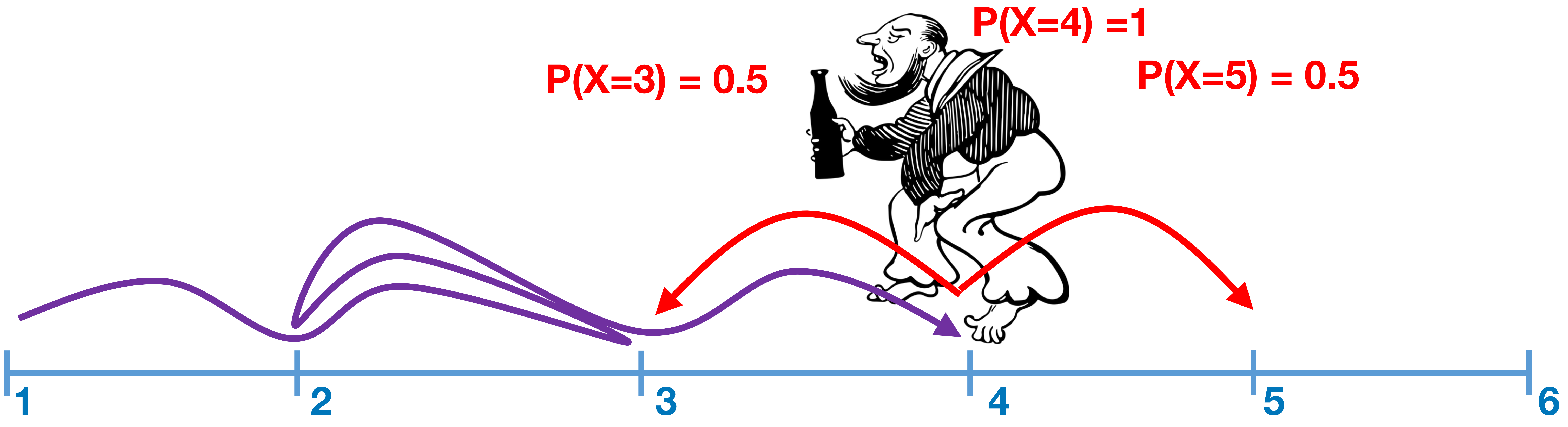
The Markov assumption postulates that past and future data are independent if one knows the current state

- A stochastic model/ process that obeys the Markov assumption is a Markov model
  - This does not mean that x_t is deterministic based on x_{t-1}
- **If** we can model our robot as a Markov process…
  - We can recursively estimate x_t using:
    - x_{t-1}, z_t, and u_t
    - **But not** $x_{0:t-1}$, $z_{1:t-1}$, and $u_{1:t}$!
    - Tractable!

Andrey Markov (1856–1922) was a Russian mathematician best known for his work on stochastic processes

Drunkards Walk

- Random walk on the number line
  - At each step, the position may change by +1 or -1 with equal probability
- The transition probabilities depend only on the current position, not on the manner in which the position was reached
- This is a Markov process!



Coin purse

- Contents
  - 5 quarters (25¢)
  - 5 dimes (10¢)
  - 5 nickels (5¢)
- Draw coins randomly, one at a time and place them on a table
- X_n is the total value of coins on the table after n draws

$$X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow$$

- Example:
  - First, I draw a nickel, what is $X_1 =$
  - Next I draw a dime, what is $X_2 =$

Coin purse

- Suppose…
  - In the first six draws, you pick all 5 nickels and 1 quarter, $X_6 = 50¢$
  - What can we say about X_7?

- Contents
  - 5 quarters (25¢)
  - 5 dimes (10¢)
  - 5 nickels (5¢)

- Is this a Markov Model?

Bayes Theorem

+

Robot-Environment Model

+

Markov Assumption

=

Bayes Filter

State Generative Model

- x_t is generated stochastically from the state x_{t-1}
- x_t depends on $x_{0:t-1}$, $z_{1:t-1}$, and $u_{1:t}$, and $p(x_t \mid x_{0:t-1}, z_{1:t-1}, u_{1:t})$ … intractable!
- If state x_t is modeled under the **Markov Assumption**, then
  - $p(x_t \mid x_{0:t-1}, z_{1:t-1}, u_{1:t}) = p(x_t \mid x_{t-1}, u_t)$
  - Knowledge of only the previous state x_{t-1} and control u_t is sufficient to predict x_t

Measurement Generative Model

- Similarly, the process by which measurements are generated are of importance
  - $p(z_t \,|\, x_{0:t-1}, z_{1:t-1}, u_{1:t})$
- If state x_t conforms to the **Markov Assumption**, then
  - $p(z_t \,|\, x_{0:t-1}, z_{1:t-1}, u_{1:t}) = p(z_t \,|\, x_t)$
  - The state x_t is sufficient to predict the (potentially noisy) measurements
  - Knowledge of any other variable, such as past measurements, controls, or even past states, is irrelevant under the Markov Assumption

Bayes Theorem

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Robot-Environment Model

+

Markov Assumption

=

Bayes Filter

Robot Belief

- Probabilistic robotics represents beliefs through *posterior conditional probability distributions*
  - Probability distributions over state variables conditioned on available data
  - The belief of a robot is the posterior distribution over the state of the environment, given all past sensor measurements and all past controls
    - Belief over a state variable x_t is denoted by $bel(x_t)$:
      - $bel(x_t) = p(x_t \,|\, z_{1:t}, u_{1:t})$
    - The (prior) belief is the belief before incorporating the latest measurement z_t:
      - $\overline{bel}(x_t) = p(x_t \,|\, z_{1:t-1}, u_{1:t})$

Bayes Filter

- A recursive algorithm that calculates the belief distribution from measurements and control data

1. **Algorithm Bayes_Filter** $(bel(x_{t-1}), u_t, z_t)$:
2. **for** all x_t do
3. $\overline{bel}(x_t) = \Sigma_{x_{t-1}}\ p(x_t \mid u_t, x_{t-1})\ bel(x_{t-1})$
4. $bel(x_t) = \eta\ p(z_t \mid x_t)\ \overline{bel}(x_t)$
5. end **for**
6. **return** $bel(x_t)$

Bayes Filter

- A recursive algorithm that calculates the belief distribution from measurements and control data

1. **Algorithm Bayes_Filter** $(bel(x_{t-1}), u_t, z_t)$:
2. $\boxed{\textbf{for all } x_t \textbf{ do}}$
3. $\boxed{\overline{bel}(x_t)} = \boxed{\Sigma_{x_{t-1}}} \boxed{\boxed{p(x_t \mid u_t, x_{t-1})} \boxed{bel(x_{t-1})}}$ **(Prediction step)**
4. $bel(x_t) = \eta \; p(z_t \mid x_t) \; \overline{bel}(x_t)$
5. end **for**
6. **return** $bel(x_t)$

Transition probability/ action model

Bayes Filter

- A recursive algorithm that calculates the belief distribution from measurements and control data

1. **Algorithm Bayes_Filter** $(bel(x_{t-1}), u_t, z_t)$:
2. $\boxed{\textbf{for all } x_t \textbf{ do}}$
3. $\overline{bel}(x_t) = \Sigma_{x_{t-1}} \, p(x_t \mid u_t, x_{t-1}) \, bel(x_{t-1})$ **(Prediction step)**
4. $\boxed{bel(x_t)} = \boxed{\eta \, \boxed{p(z_t \mid x_t)} \, \overline{bel}(x_t)}$ **(Update/measurement step)**
5. end **for**
6. $\boxed{\textbf{return } bel(x_t)}$

Transition probability/ action model

Measurement probability/ sensor model

Kalman Filter Implementation

Kalman Filter ($\mu(t-1)$, $\Sigma(t-1)$, $u(t)$, $z(t)$)

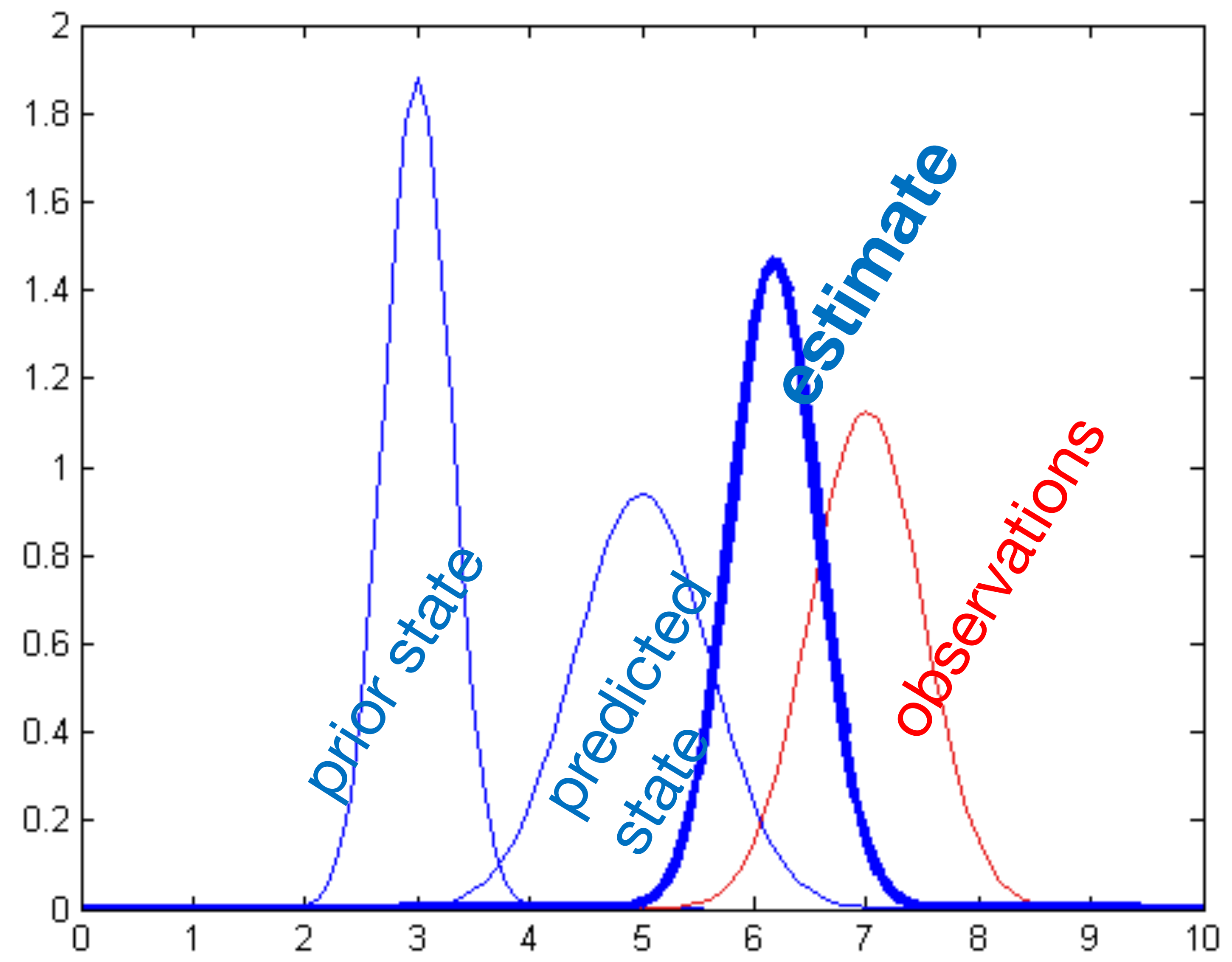
prediction

1. $\mu_p(t) = A\mu(t-1) + Bu(t)$
2. $\Sigma_p(t) = A\Sigma(t-1)A^T + \Sigma_u$

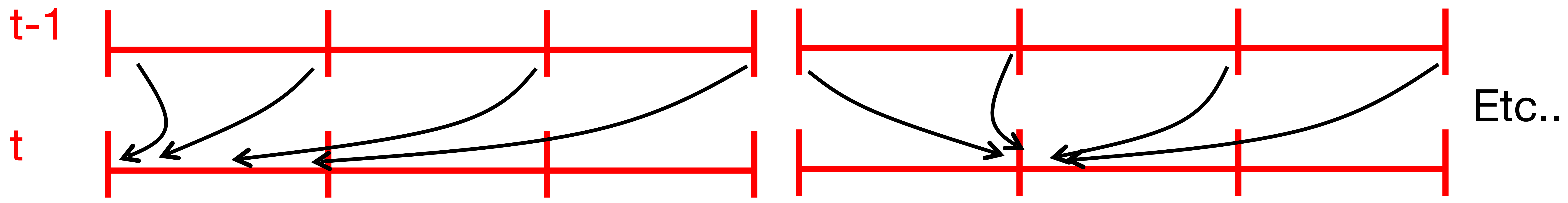
update

3. $K_{KF} = \Sigma_p(t)C^T(C\Sigma_p(t)C^T + \Sigma_z)^{-1}$
4. $\mu(t) = \mu_p(t) + K_{KF}(z(t) - C\mu_p(t))$
5. $\Sigma(t) = (I - K_{KF}C)\Sigma_p(t)$
6. Return $\mu(t)$ and $\Sigma(t)$

State estimate: $\mu(t)$
State uncertainty: $\Sigma(t)$
Process noise: Σ_u
Kalman filter gain: K_{KF}
Measurement noise: Σ_z



Bayes Filter



1. **Algorithm Bayes_Filter** $(bel(x_{t-1}), u_t, z_t)$:
2. **for** all x_t do
3. $\overline{bel}(x_t) = \Sigma_{x_{t-1}} \; p(x_t \mid u_t, x_{t-1}) \; bel(x_{t-1})$ **(Prediction step)**
4. $bel(x_t) = \eta \; p(z_t \mid x_t) \; \overline{bel}(x_t)$ **(Update/measurement step)**
5. end **for**
6. **return** $bel(x_t)$

Transition probability/ action model

Measurement probability/ sensor model

Bayes Filter

Dynamic stochastic model

- $p(x_t \mid x_{t-1}, u_t)$ is the **state transition probability**
  - How the robot state x_t evolves over time as a function of the control u_t
- $p(z_t \mid x_t)$ is the **measurement probability**
  - How measurements are generated from the robot state x_t
  - Informally, you can think of measurements as noisy projections of x_t
- Remember that these prediction are *stochastic and not deterministic*

Bayes Filter

Initial conditions

- To compute the posterior belief recursively, the algorithm requires an initial belief $bel(x_0)$ at time $t = 0$

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Bayes Filter

Initial conditions

- To compute the posterior belief recursively, the algorithm requires an initial belief $bel(x_0)$ at time $t = 0$
- If we know the initial state with absolute certainty, we can initialize a point mass distribution that centers all probability mass on the correct value of x_0 and assign zero everywhere else
- If we are entirely ignorant of the initial state, we can initialize it with a uniform probability distribution over all the possible states

References

1. Thrun, Sebastian, Wolfram Burgard, and Dieter Fox. Probabilistic robotics. MIT press, 2005.
2. http://gki.informatik.uni-freiburg.de/teaching/ws0607/advanced/recordings/BayesFiltering.pdf
3. http://home.deib.polimi.it/restelli/MyWebSite/pdf/E05_h.pdf
4. https://en.wikipedia.org/wiki/Markov_chain
5. https://en.wikipedia.org/wiki/Markov_model
6. Image Sources:
   a. https://blog.robotiq.com/a-brief-history-of-robots-in-manufacturing
   b. https://northwesturbanist.files.wordpress.com/2015/01/20151104_082523.jpg
   https://expresswriters.com/is-the-dress-blue-black-white-or-gold-how-it-went-viral/
   https://en.wikipedia.org/wiki/Andrey_Markov