Linear Systems Recap

Fast Robots, ECE4160/5160, MAE 4190/5190

E. Farrell Helbling, 2/20/25

Class Action Items

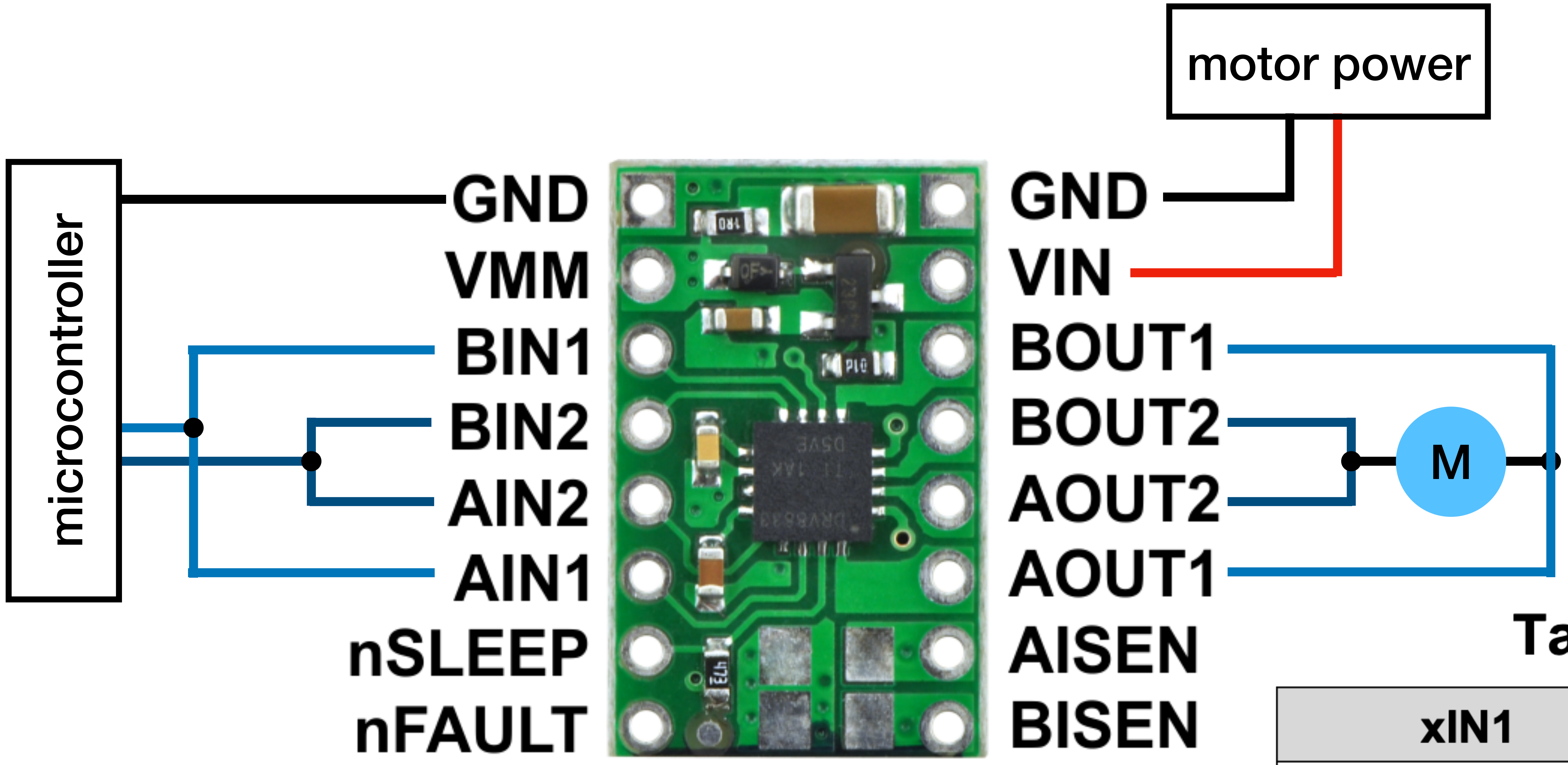
- Lab 3 is due Feb 25-26, if you need to use a slip week, please send us a private message on Ed. You can do this up until the deadline.
- Lab 4 starts next week, if you want to get a head start during open hours, we will discuss the lab in class today and the website is already posted.
  - Lab 4 has another soldering component, so think about how you want to connect things ahead of time!
  - At the end of Lab 4, you will have a fully-integrated RC car.
  - Good example from last year: https://nila-n.github.io/Lab4.html
- Please respond to the Ed Discussion polls for workload!

Lab 4 Open loop control

https://nila-n.github.io/Lab4.html

Brushed DC motor controllers

Parallel-coupled motor controller



Fast decay: “coasting”

Slow decay: “braking”

Table 2. PWM Control of Motor Speed

xIN1	xIN2	FUNCTION
PWM	0	Forward PWM, fast decay
1	PWM	Forward PWM, slow decay
0	PWM	Reverse PWM, fast decay
PWM	1	Reverse PWM, slow decay

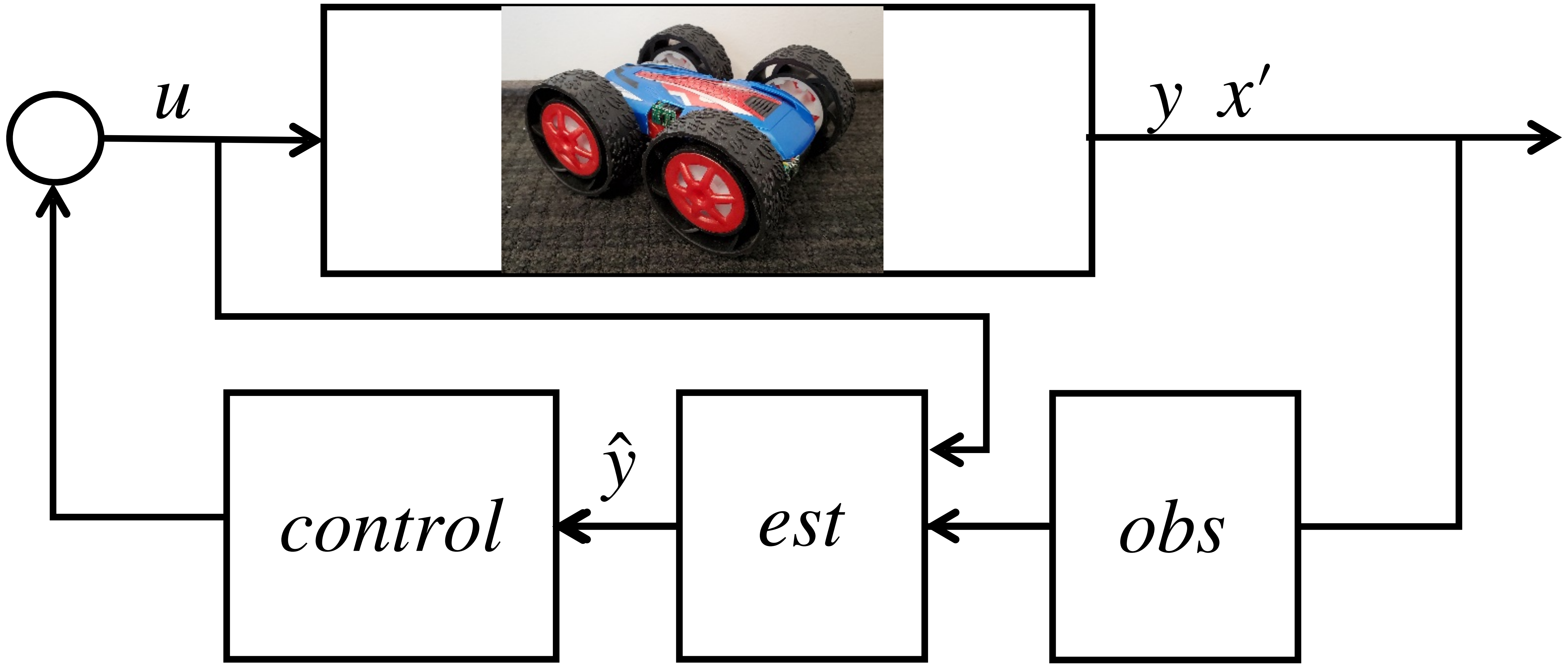
https://www.pololu.com/product-info-merged/2130

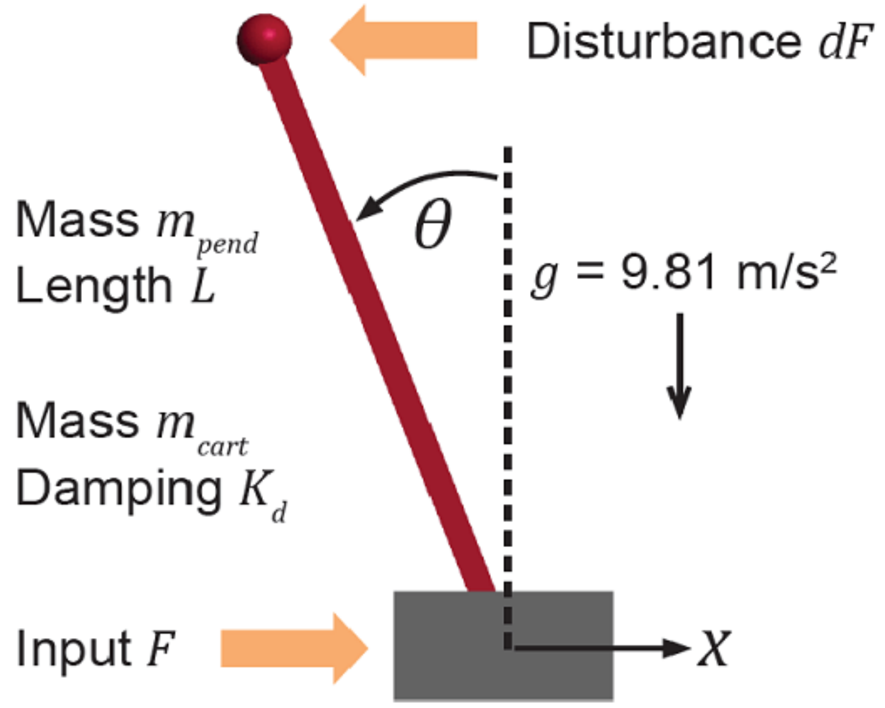
Linear Systems Recap

Linear Systems

- Linear systems review
- Eigenvectors and eigenvalues
- Stability
- Discrete time systems
- Linearizing nonlinear systems
- Controllability
- Observability

$$\dot{x} = Ax + Bu$$





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$$\dot{x} = Ax + Bu$$

These should look familiar from:

- MATH2940 Linear Algebra
- ECE3250 Signals and Systems
- ECE5210 Theory of Linear Systems
- MAE3260 System Dynamics
- and many others…

Based on "Control Bootcamp", Steve Brunton, UW
https://www.youtube.com/watch?v=Pi7l8mMjYVE

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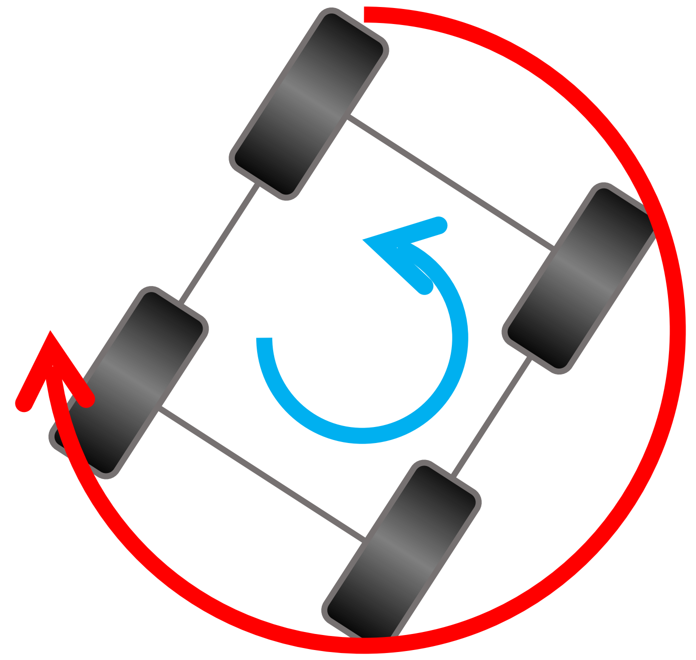

- 1st order system: $\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & \frac{-c}{I} \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{I} \end{bmatrix} u$
- 2nd order system: $\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \text{const} & \frac{-c}{I} \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{I} \end{bmatrix} u$

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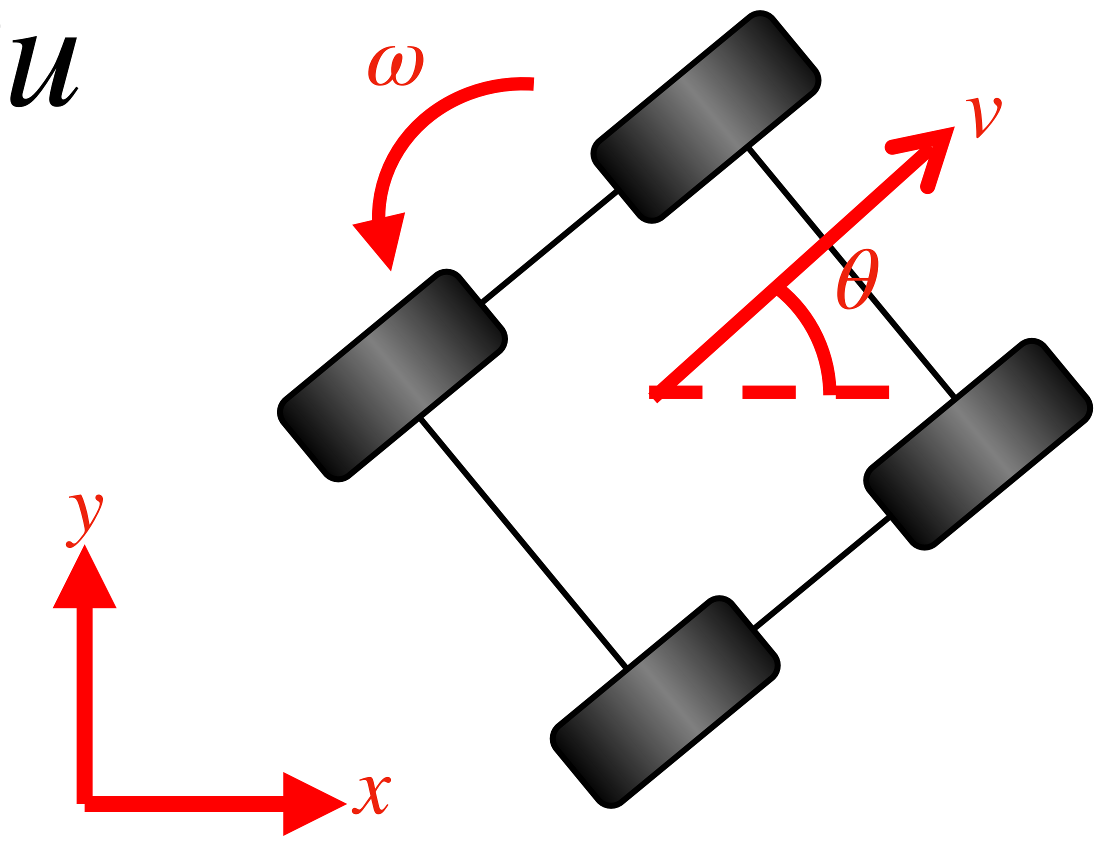

$$\dot{x} = \cos(\theta)v$$
$$\dot{y} = \sin(\theta)v$$
$$\dot{\theta} = \omega$$

$$\boxed{\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos(\theta) & 0 \\ \sin(\theta) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}}$$

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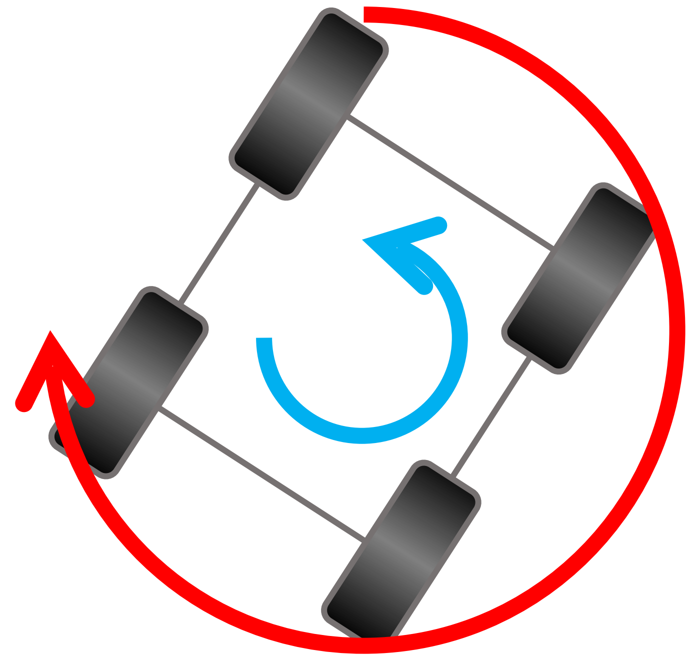
$$\dot{x} = Ax + Bu$$



- 1st order system: $\begin{bmatrix}\dot{\theta}\\ \ddot{\theta}\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & \frac{-c}{I}\end{bmatrix}\begin{bmatrix}\theta\\ \dot{\theta}\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{I}\end{bmatrix}u$
- 2nd order system: $\begin{bmatrix}\dot{\theta}\\ \ddot{\theta}\end{bmatrix} = \begin{bmatrix}0 & 1\\ \text{const} & \frac{-c}{I}\end{bmatrix}\begin{bmatrix}\theta\\ \dot{\theta}\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{I}\end{bmatrix}u$

Linear Systems

- Linear system $\dot{x} = Ax$ $\quad x \in \mathbb{R}^n \quad A \in \mathbb{R}^{n \times n}$
- Basic solution $x(t) = e^{At}x(0)$

Aside: $\dfrac{dx}{dt} = kx \leftrightarrow \dfrac{dx}{x} = kdt \leftrightarrow \ln(|x|) = kt + c$

$|x| = e^{kt} + e^{c} \leftrightarrow x = \pm ce^{kt}$

- Taylor series expansion $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots$

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \ldots$$

Linear Systems

- Linear system $\dot{x} = Ax$
- Basic solution $x(t) = e^{At}x(0)$
- Map the system to eigenvector coordinates to make computation easier
  - Apply a linear transform: $z = Tx \quad \leftrightarrow x = T^{-1}z$
  - Substitute into the original equation: $T^{-1}\dot{z} = AT^{-1}z \;\leftrightarrow z = TAT^{-1}z$
  - Pick the matrix, T, such that TAT^{-1} becomes simpler than A

Eigenvectors and Eigenvalues

Eigenvectors and Eigenvalues

- Eigenvectors, ξ, of A $\qquad A\xi = \lambda\xi$
- Matrix of eigenvectors, T

$$T = \begin{bmatrix} \xi_1 & \xi_2 & \dots & \xi_n \end{bmatrix}$$

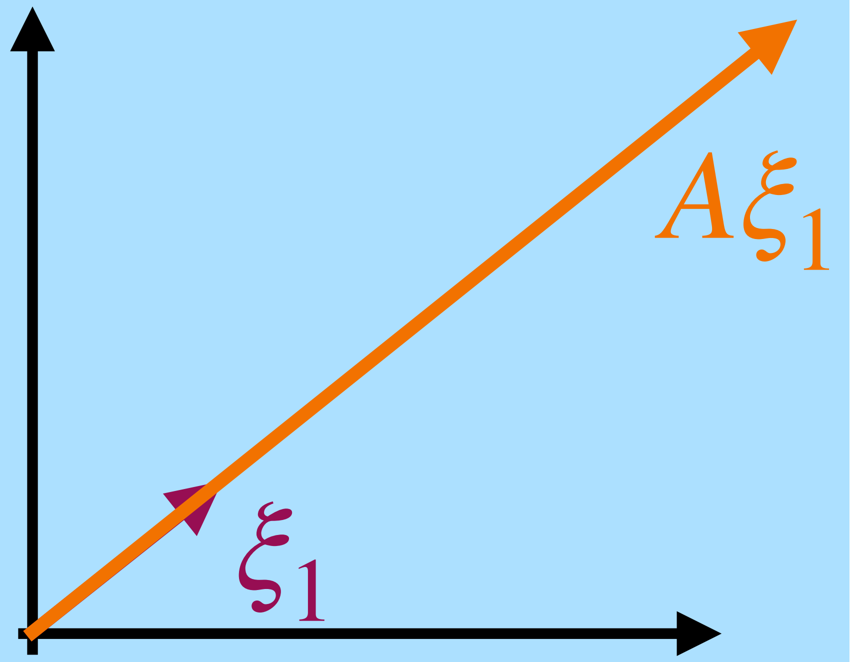
- Diagonal matrix of eigenvalues, D

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$$\boxed{AT = TD}$$

$$A = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$$

$$\xi_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$



$$\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 8 \end{bmatrix} = 4 \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\lambda_1 = 4$$

z-coordinates

$$\dot{x} = Ax \qquad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$x(t) = e^{At}x(0)$$

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \ldots$$

$$AT = TD \qquad \leftrightarrow T^{-1}AT = D$$

$$x = Tz$$

$$\dot{x} = T\dot{z} = Ax$$

$$T\dot{z} = ATz$$

$$\dot{z} = T^{-1}ATz$$

$$\boxed{\dot{z} = Dz}$$

MATLAB has a handy function: [T, D] = eig(A);

By mapping our system to eigenvector coordinates, the dynamics become diagonal (very simple!)

z-coordinates

$$\dot{x} = Ax = T\dot{z}$$

$$x(t) = e^{At}x(0)$$

$$T^{-1}AT = D$$

$$\dot{z} = Dz$$

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

$$z_1(t) = e^{\lambda_1 t} z_1(0) \quad \ldots \quad z_n(t) = e^{\lambda_n t} z_n(0)$$

$$z(t) = e^{Dt}z(0) = \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix} z_0$$

Much simpler to think about the system in eigen coordinates!

Let’s get back to x-coordinates

$$\boxed{\begin{aligned} &\dot{x} = Ax = T\dot{z} \\ &x(t) = e^{At}x(0) \\ &T^{-1}AT = D \\ &\dot{z} = Dz \end{aligned}}$$

$$A^n = TD^nT^{-1}$$

$$x(t) = e^{At}x(0)$$

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots$$

$$e^{At} = e^{TDT^{-1}t}$$

$$e^{At} = I + TDT^{-1}t + \underbrace{\left(TD\underbrace{T^{-1}T}_{I}DT^{-1}\right)}_{\left(TD^2T^{-1}\right)}\frac{t^2}{2!} + \dots$$

Let's get back to x-coordinates

$$\dot{x} = Ax = T\dot{z}$$

$$x(t) = e^{At}x(0)$$

$$T^{-1}AT = D$$

$$\dot{z} = Dz$$

$$A^n = TD^nT^{-1}$$

$$x(t) = e^{At}x(0)$$

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \ldots$$

$$e^{At} = e^{TDT^{-1}t}$$

$$e^{At} = I + TDT^{-1}t + \left(TDT^{-1}TDT^{-1}\right)\frac{t^2}{2!} + \ldots$$

$$e^{At} = T\left[I + Dt + \frac{D^2t^2}{2!} + \ldots + \frac{D^nt^n}{n!}\right]T^{-1}$$

Easy to compute!

Let's get back to x-coordinates

$$\dot{x} = Ax = T\dot{z}$$

$$x(t) = e^{At}x(0)$$

$$AD = TD$$

$$x = Tz$$

$$e^{At} = Te^{Dt}T^{-1}$$

System solution in physical coordinates

$$x(t) = Te^{Dt}T^{-1}x(0)$$

$$z(0)$$

$$z(t)$$

$$x(t)$$

Eigenvalues and Stability

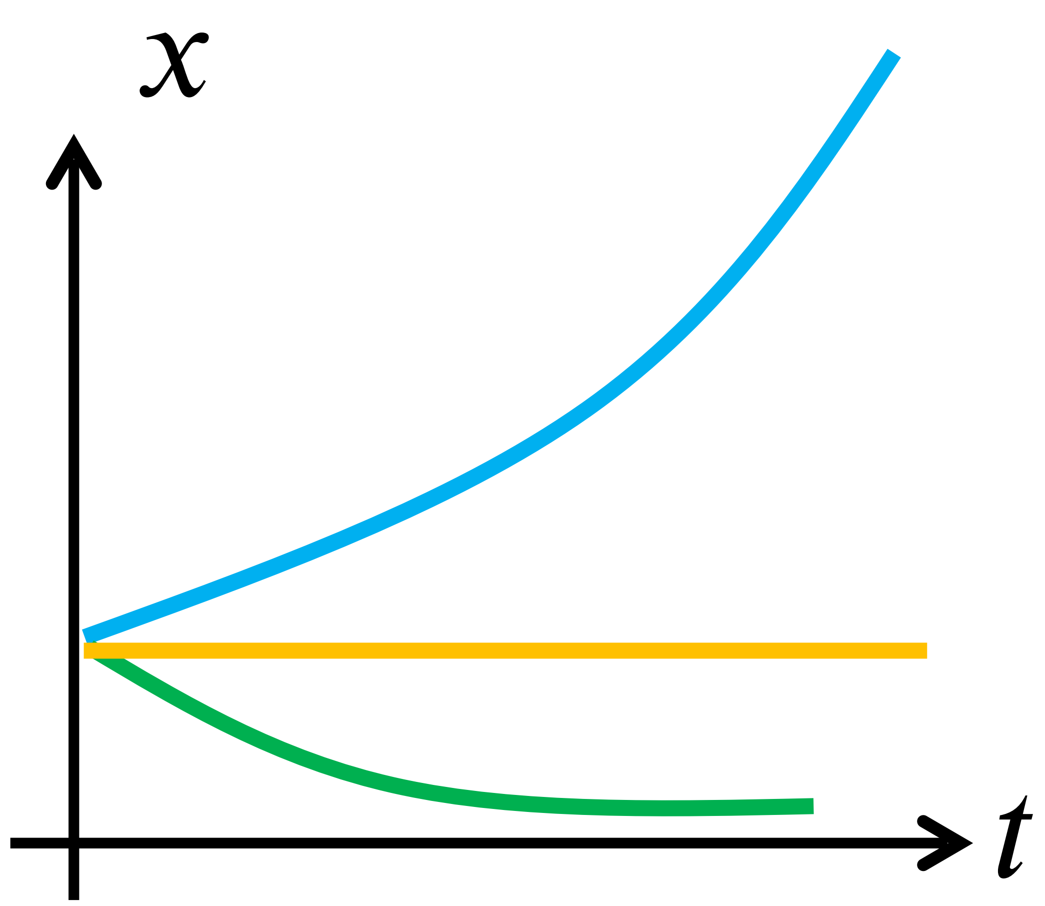
Stability (continuous time)

$$\dot{x} = Ax$$

$$x(t) = Te^{Dt}T^{-1}x(0)$$

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \quad e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix}$$

Python: eigenvalues, eigenvectors = np.linalg.eig(A)



- If even one of the $e^{\lambda t}$ goes to ∞, all go to ∞
- Complex eigenvalues: $\lambda = a + ib$
- Euler's formula: $e^{\lambda t} = e^{at}(\cos(bt) + i\sin(bt))$

Stability (continuous time)

$$\dot{x} = Ax$$

$$x(t) = Te^{Dt}T^{-1}x(0)$$

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \quad e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix}$$

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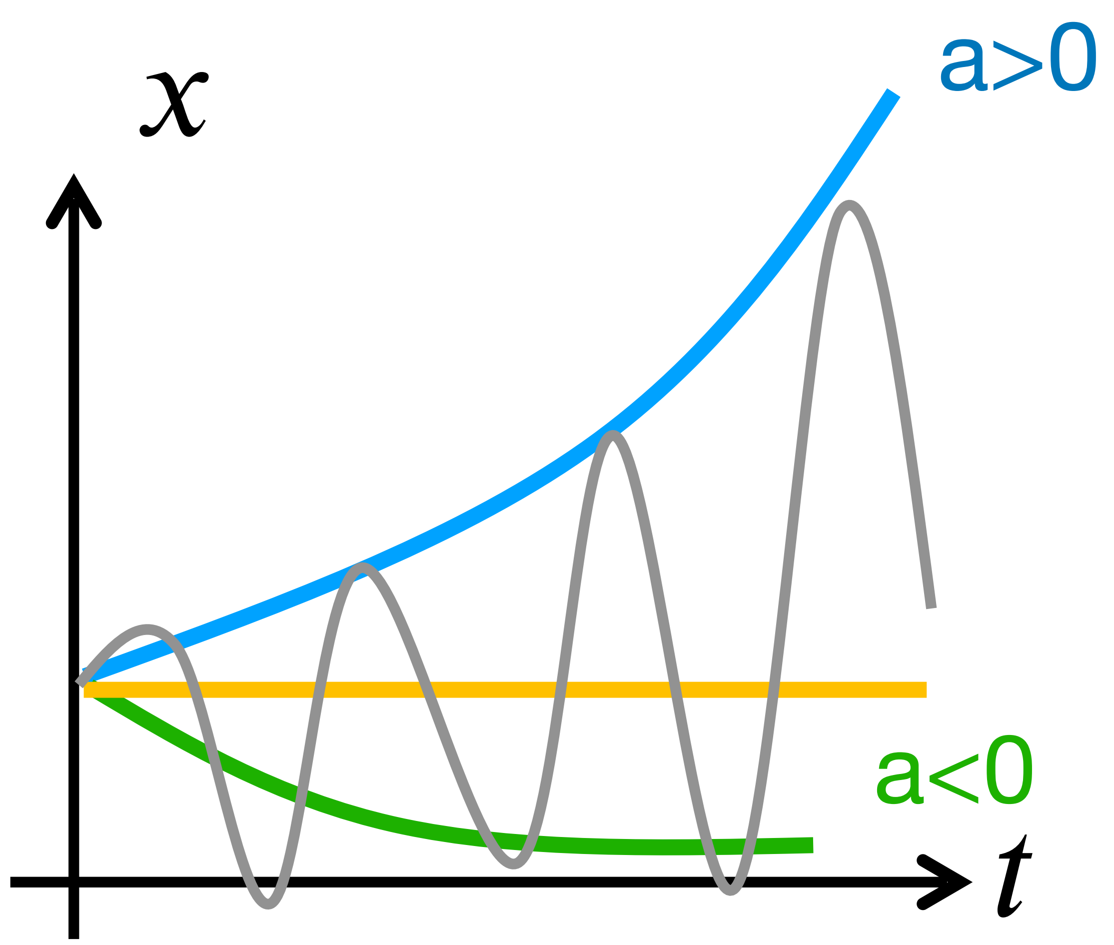


- If even one of the $e^{\lambda t}$ goes to ∞, all go to ∞
- Complex eigenvalues: $\lambda = a \pm ib$
- Euler's formula: $e^{\pm\lambda t} = e^{at}\overbrace{(\cos(bt) \pm i\sin(bt))}^{=1}$

System is stable iff real parts of all eigenvalues are <0!

Stability (continuous time)

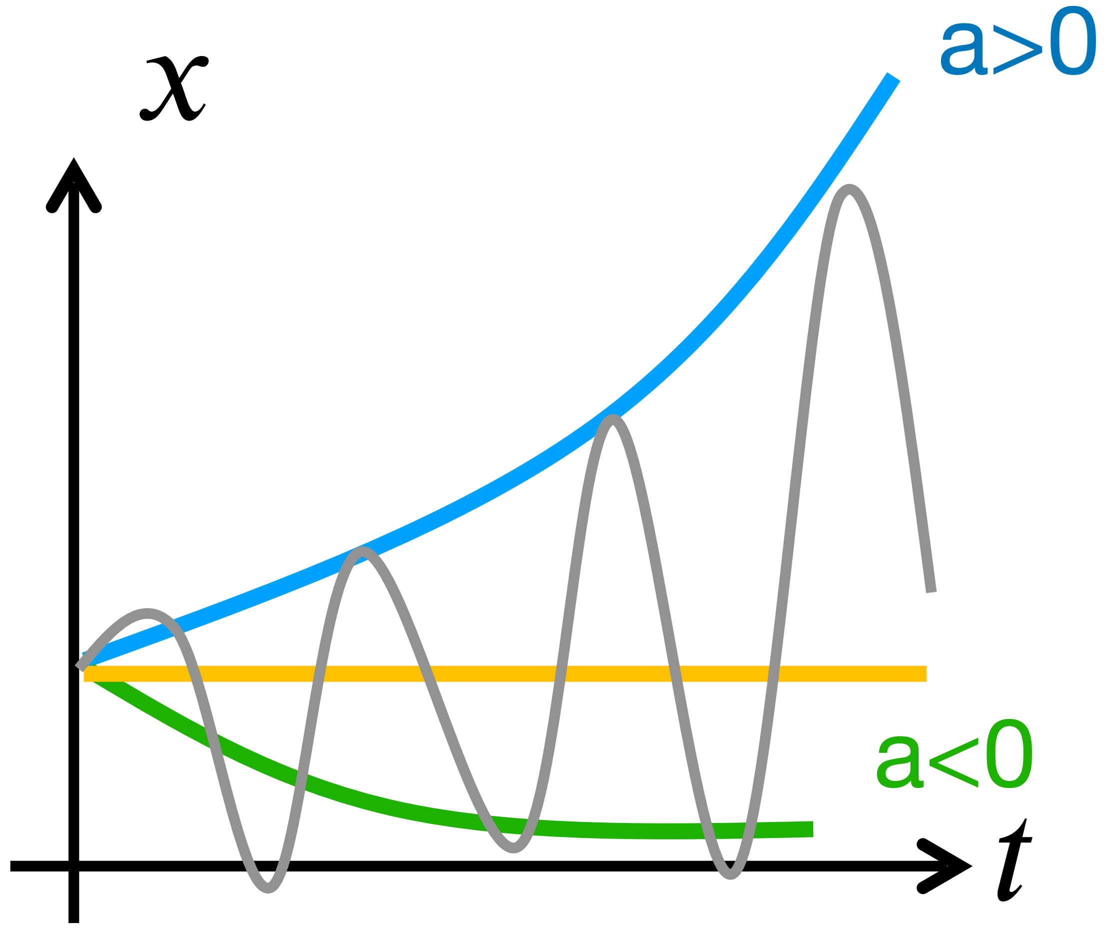
$$\dot{x} = Ax$$

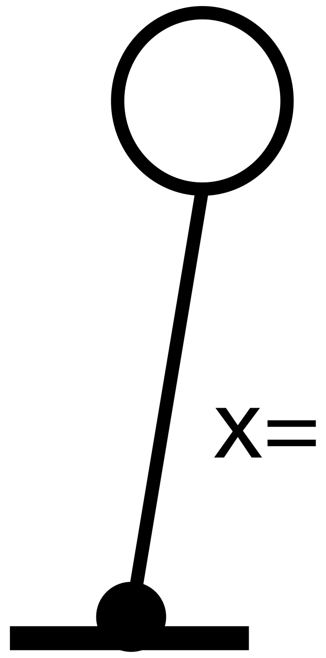
$$x(t) = Te^{Dt}T^{-1}x(0)$$

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \quad e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix}$$

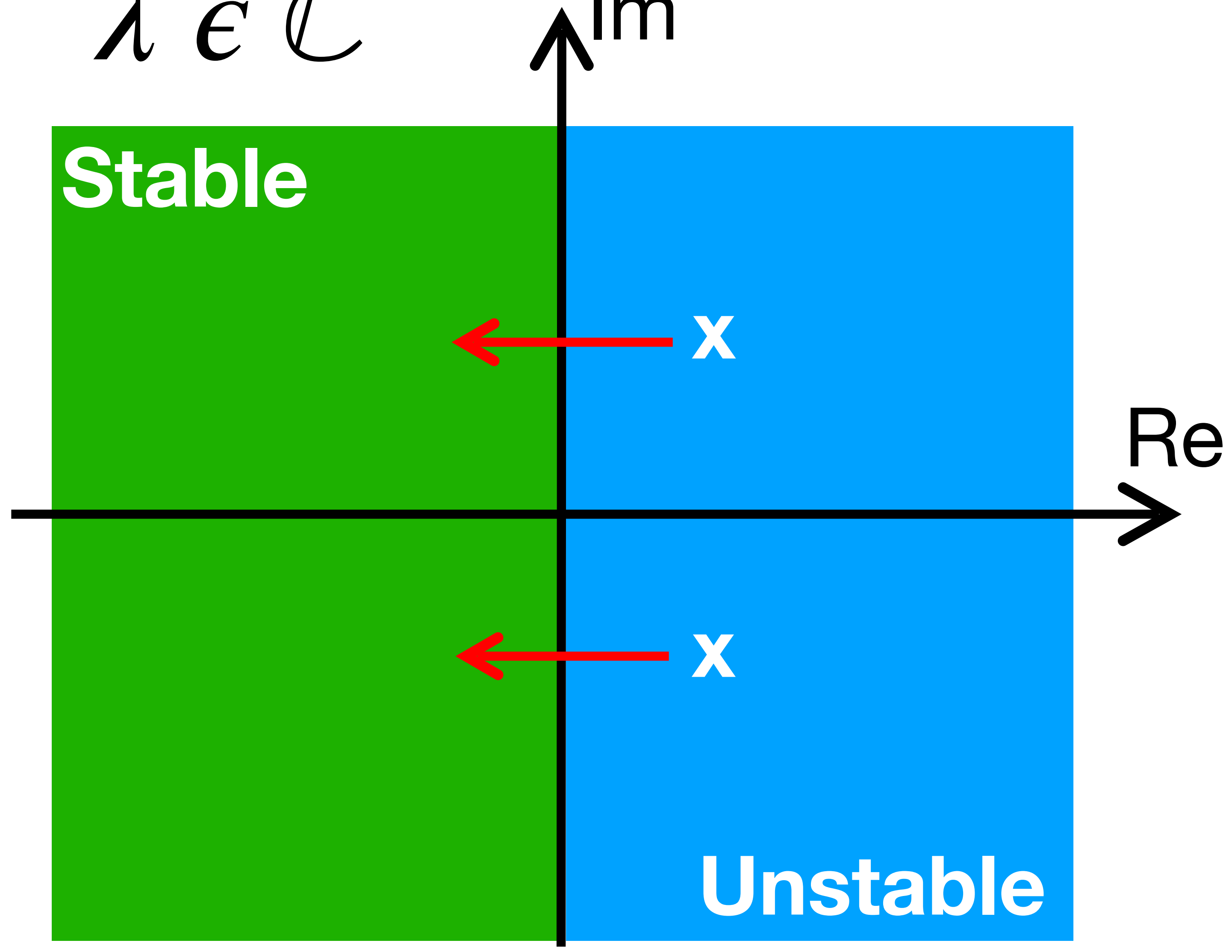
$$\lambda = a + ib$$

$$\lambda \in \mathbb{C}$$





$$x = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$



System is stable iff real parts of all eigenvalues are <0!

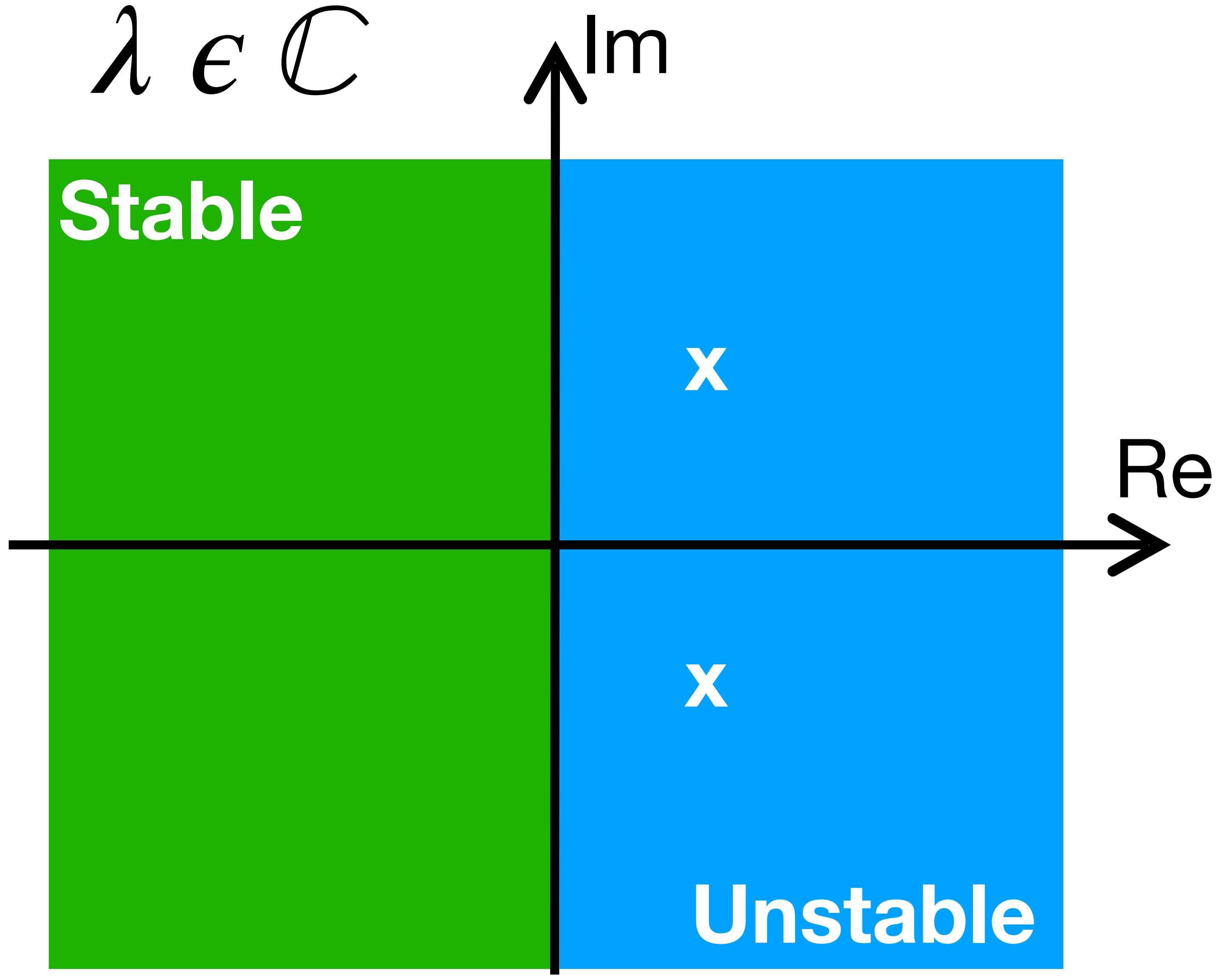
Discrete Time Systems

Stability (discrete time)

$$\dot{x} = Ax$$

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$\lambda \in \mathbb{C}$



$x(k+1) = \tilde{A}x(k)$, where $x(k) = x(k\Delta t)$

How does $\tilde{A}$ relate to A? $\tilde{A} = e^{A\Delta t}$

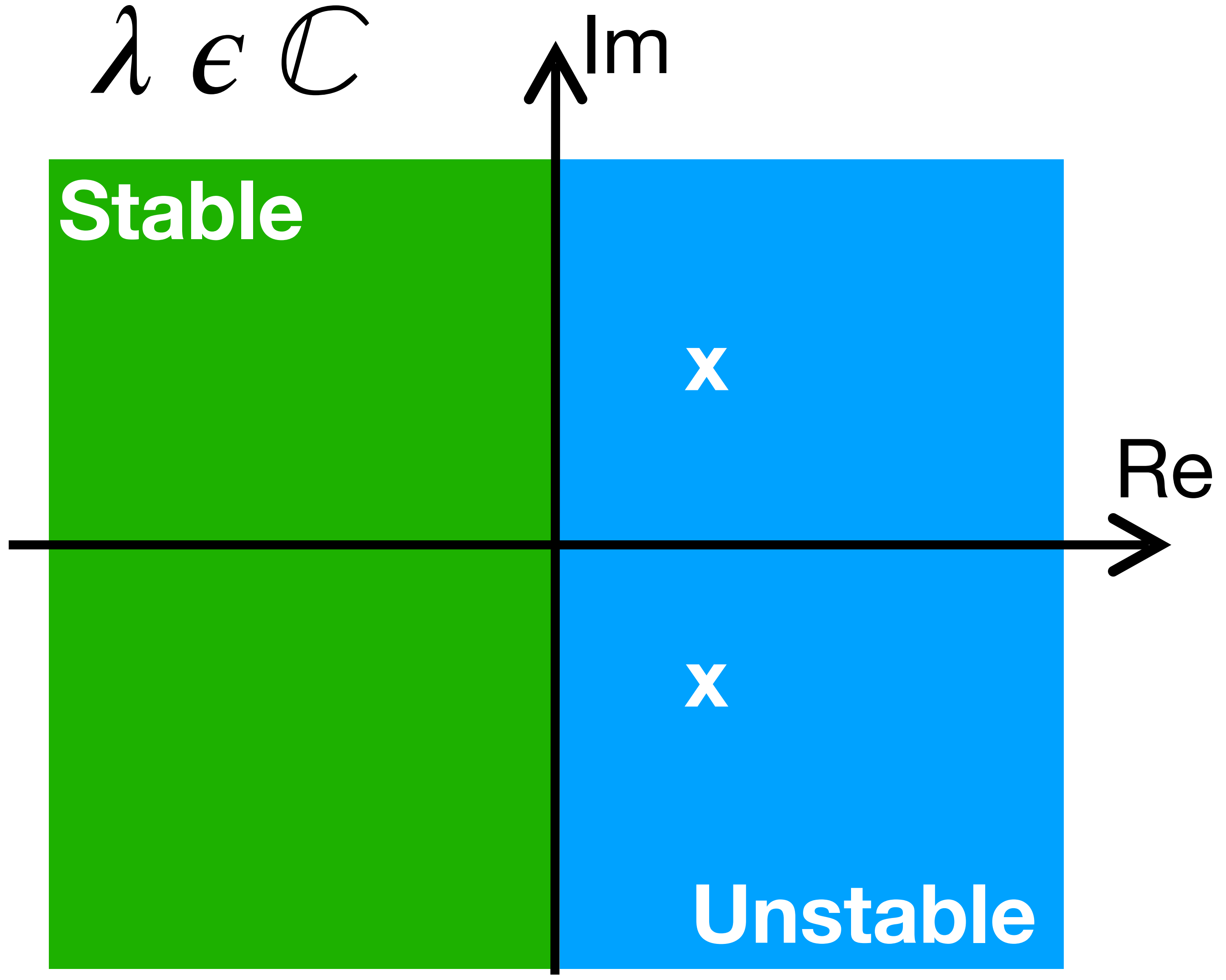
$$\begin{array}{lll} x_1 = \tilde{A}x_0 & \tilde{A} = \tilde{T}\tilde{D}\tilde{T}^{-1} & \tilde{\lambda} \\ x_2 = \tilde{A}x_1 = \tilde{A}^2 x_0 & \tilde{A}^2 = \tilde{T}\tilde{D}^2\tilde{T}^{-1} & \tilde{\lambda}^2 \\ x_3 = \tilde{A}^3 x_0 & \tilde{A}^3 = \tilde{T}\tilde{D}^3\tilde{T}^{-1} & \tilde{\lambda}^3 \\ \vdots & \vdots & \vdots \\ x_n = \tilde{A}^n x_0 & \tilde{A}^n = \tilde{T}\tilde{D}^n\tilde{T}^{-1} & \tilde{\lambda}^n \end{array}$$

Stability (discrete time)

$\dot{x} = Ax$

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$\lambda \in \mathbb{C}$



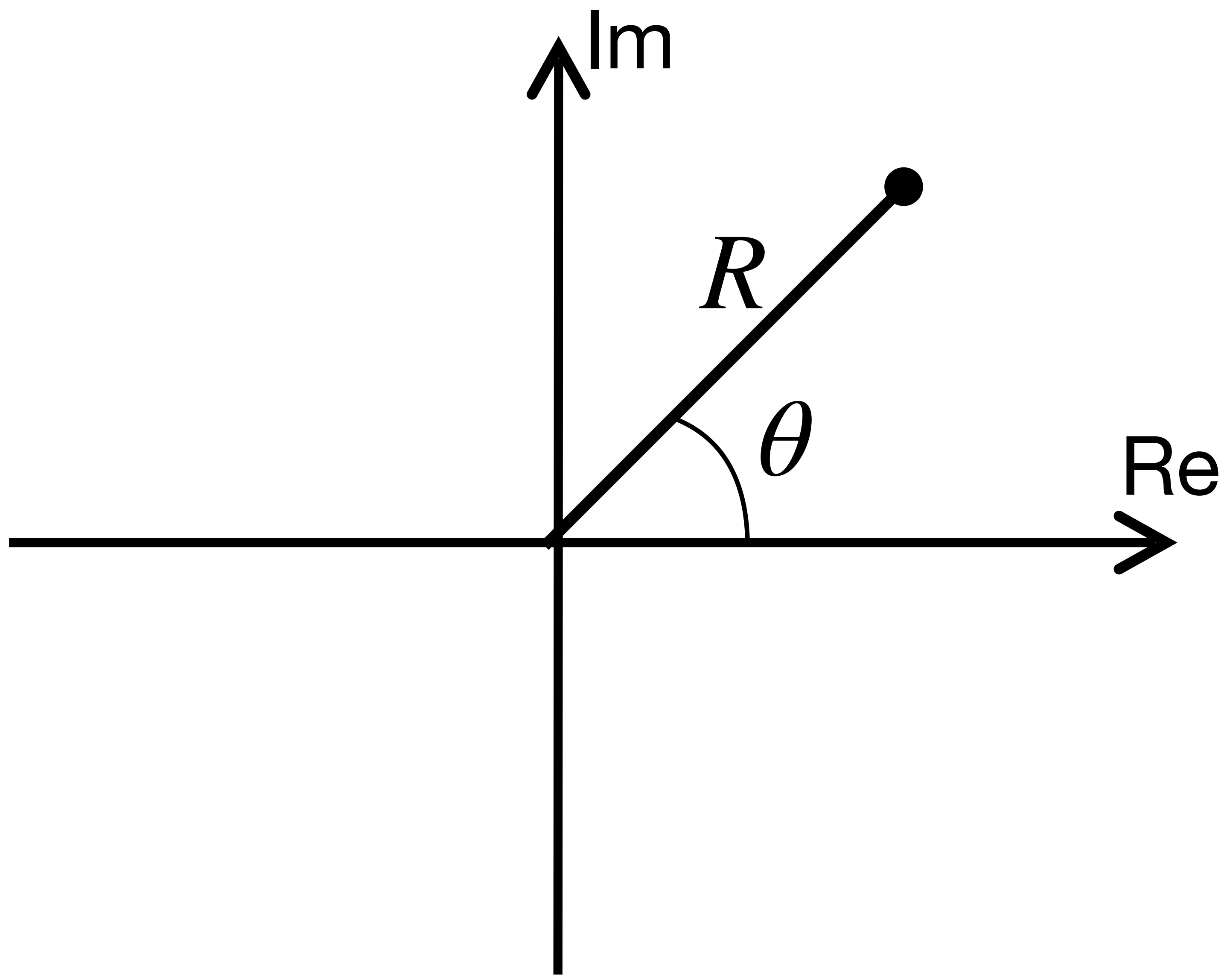
$x(k+1) = \tilde{A}x(k)$, where $x(k) = x(k\Delta t)$

How does $\tilde{A}$ relate to A? $\tilde{A} = e^{A\Delta t}$

$x_n = \tilde{A}^n x_0$ $\quad \tilde{A}^n = \tilde{T}\tilde{D}^n\tilde{T}^{-1}$ $\quad \tilde{\lambda}^n$

$\tilde{\lambda} = Re^{i\theta}$

$\tilde{\lambda}^n = R^n e^{in\theta}$

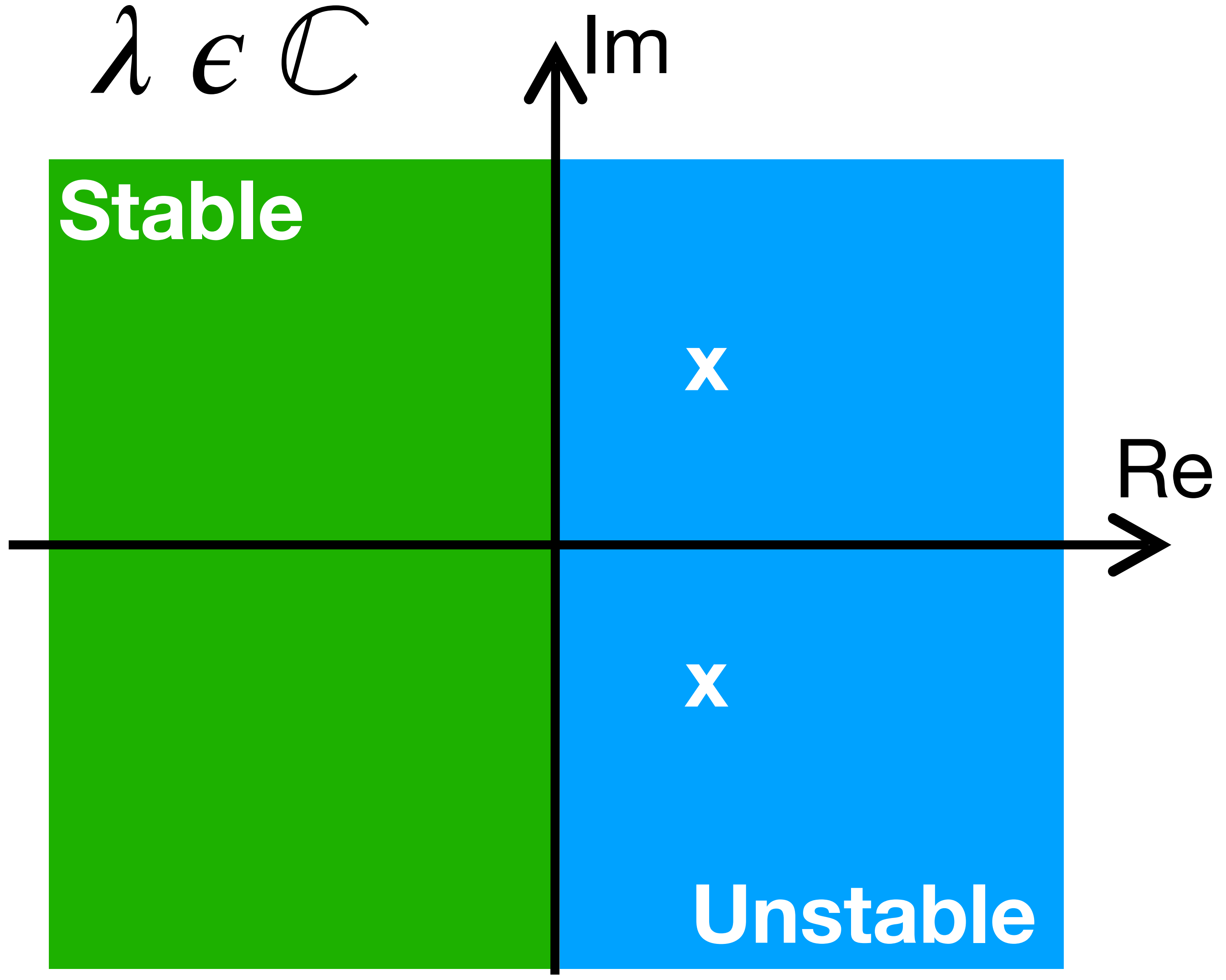


Stability (discrete time)

$\dot{x} = Ax$

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$\lambda \in \mathbb{C}$



$x(k+1) = \tilde{A}x(k)$, where $x(k) = x(k\Delta t)$

How does $\tilde{A}$ relate to A? $\quad \tilde{A} = e^{A\Delta t}$

$x_n = \tilde{A}^n x_0 \qquad \tilde{A}^n = \tilde{T}\tilde{D}^n\tilde{T}^{-1} \qquad \tilde{\lambda}^n$

$\tilde{\lambda} = Re^{i\theta}$

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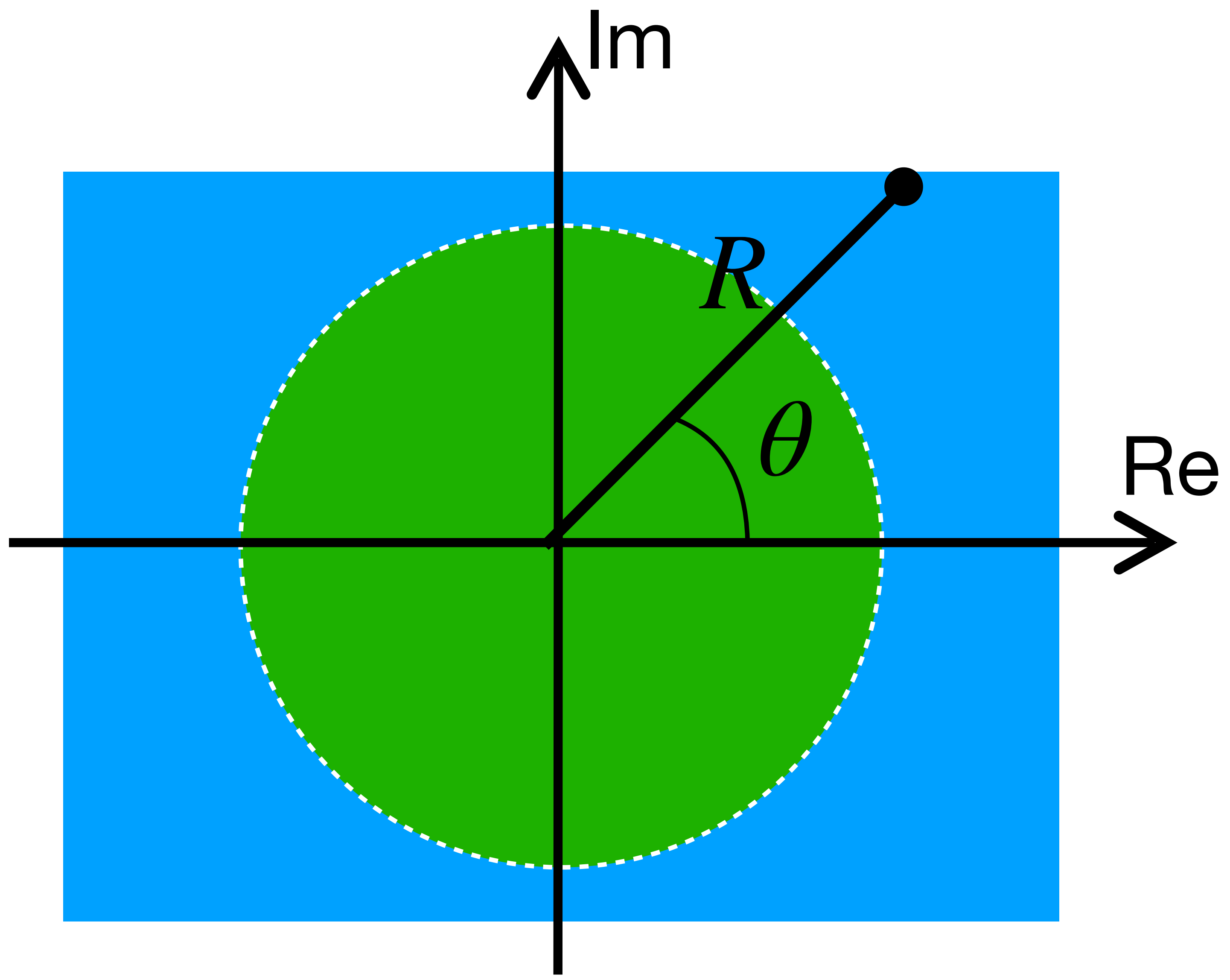

Stability (discrete time)

$$\dot{x} = Ax$$

$$x(k+1) = \tilde{A}x(k), \text{ where } x(k) = x(k\Delta t)$$

How does $\tilde{A}$ relate to A? $\quad \tilde{A} = e^{A\Delta t}$

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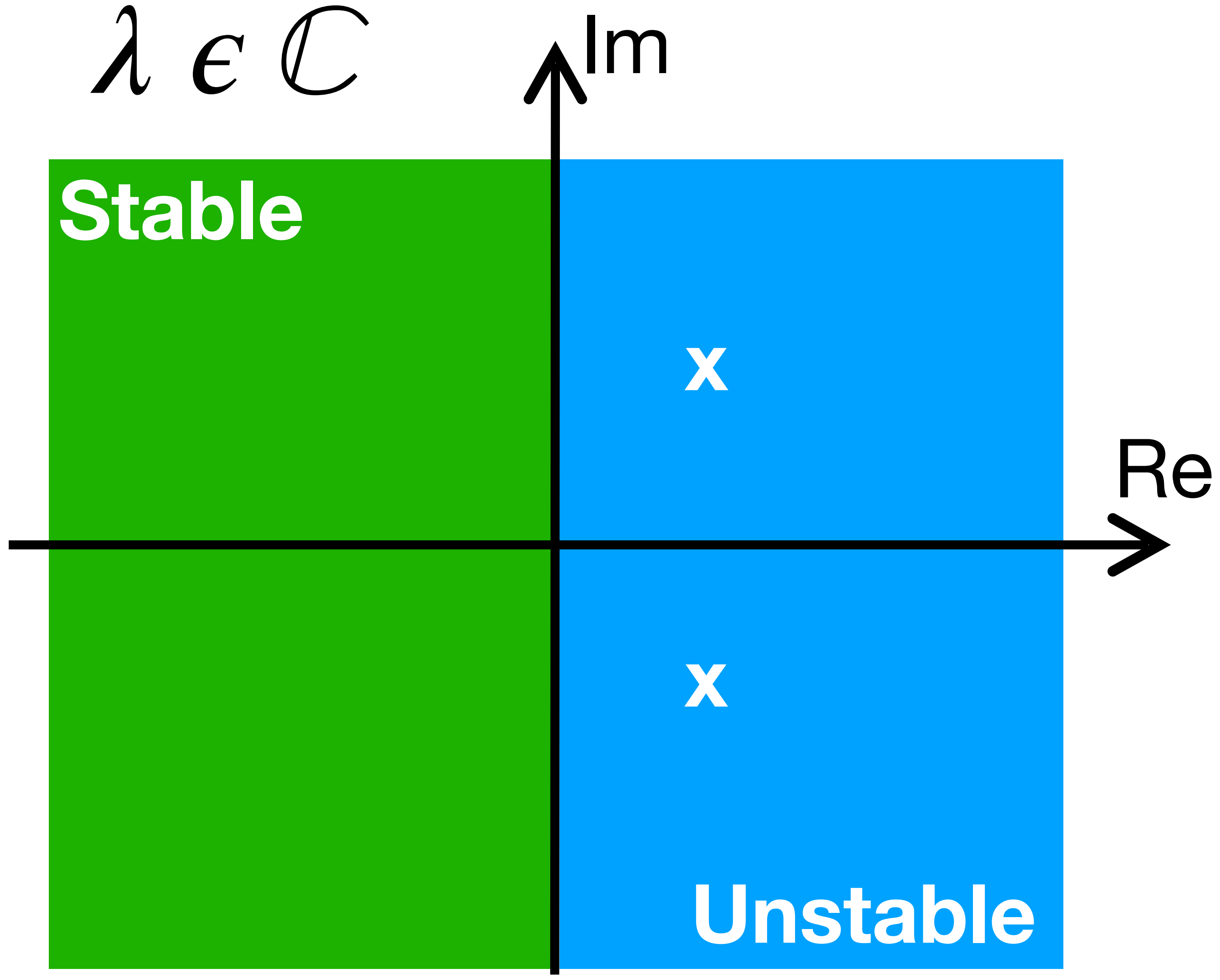
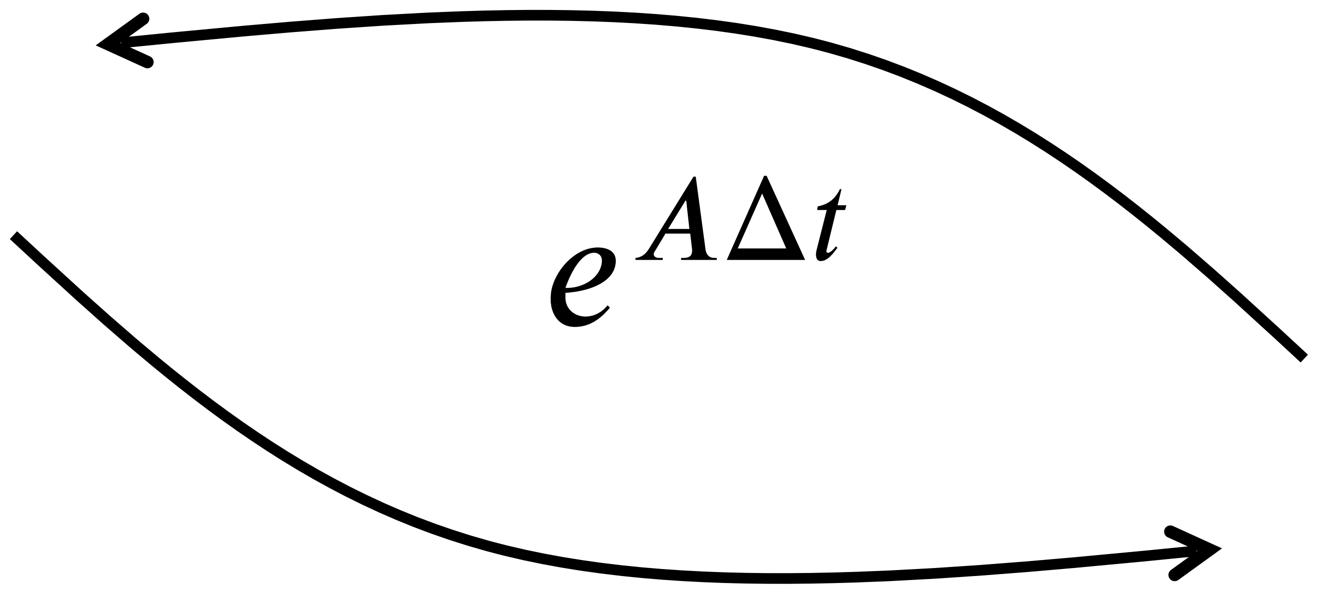
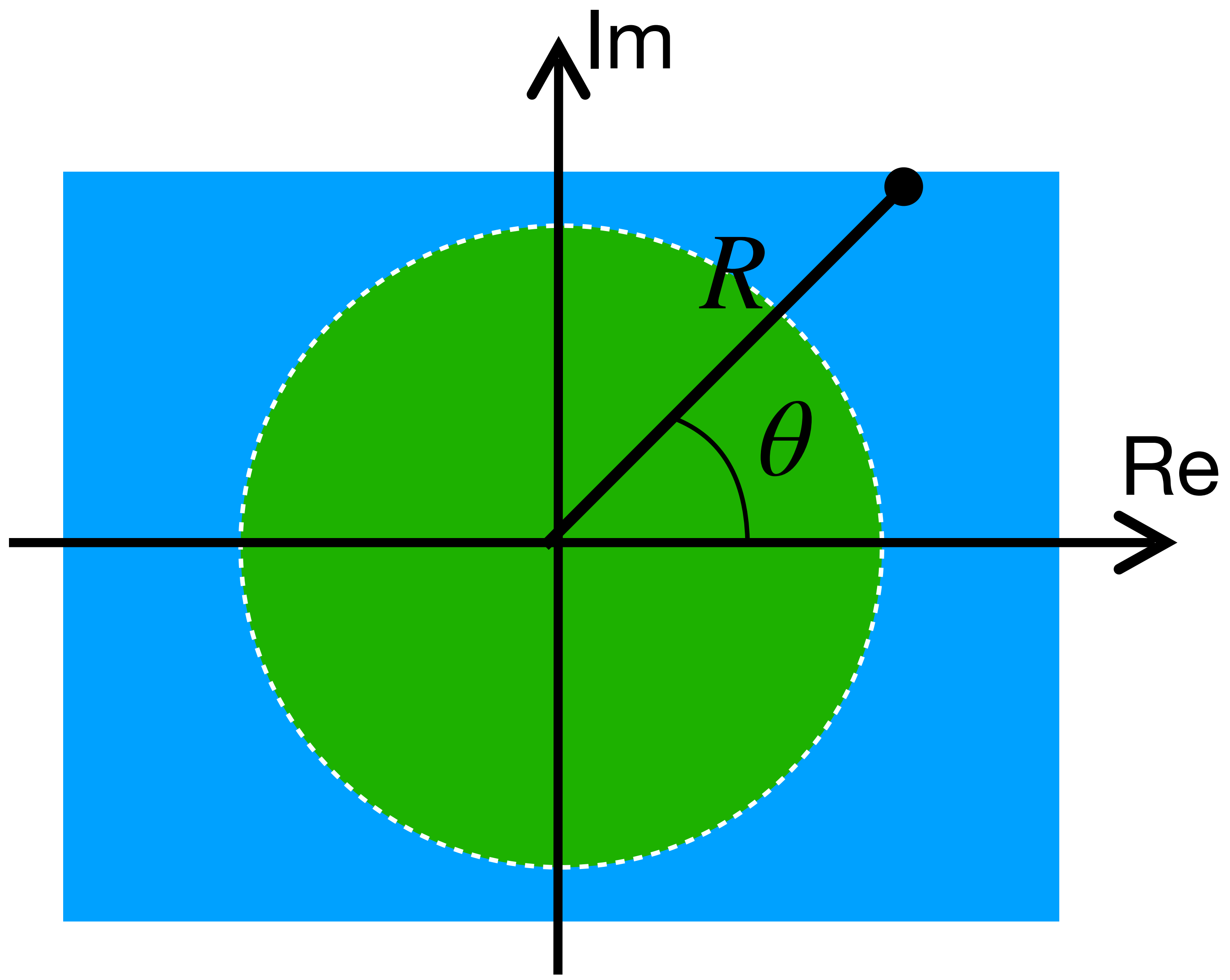
$$\tilde{\lambda} = Re^{i\theta}$$

$$\tilde{\lambda}^n = R^n e^{i\theta}$$

$\lambda \in \mathbb{C}$







Stability (discrete time)

$$\dot{x} = Ax$$

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

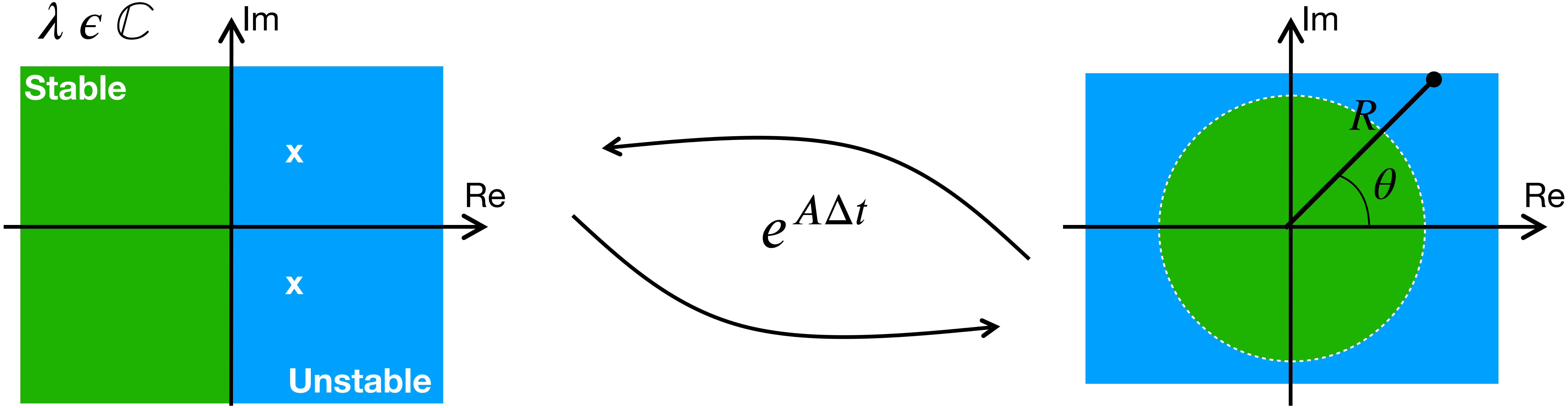
$$x(k+1) = \tilde{A}x(k)$$

$$\tilde{A} = e^{A\Delta t}$$

$$\tilde{\lambda}^n = R^n e^{i\theta}$$

- We often work in discrete time
- Stability and quality of controllers depend on sampling rate

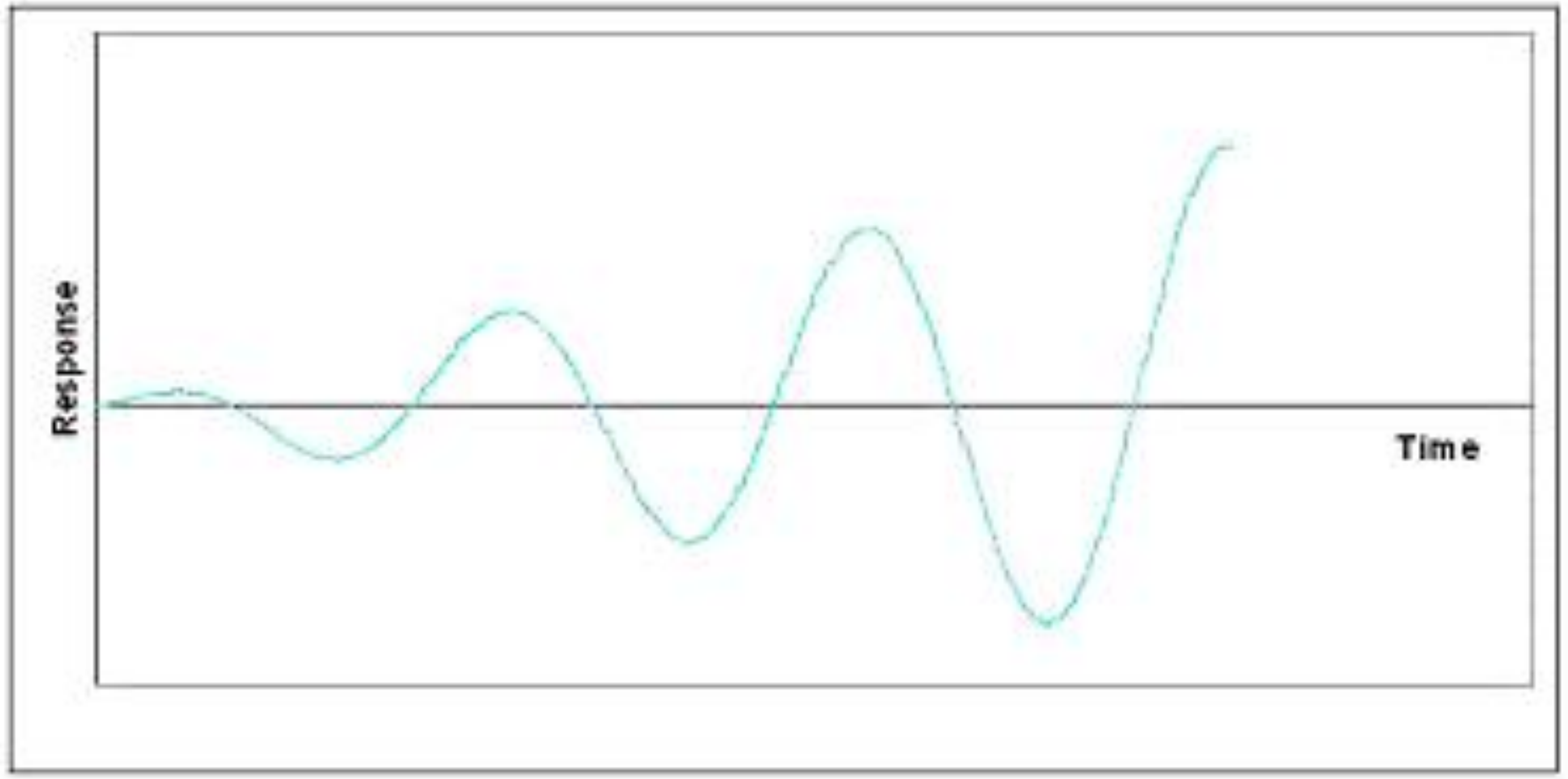
$\lambda \in \mathbb{C}$

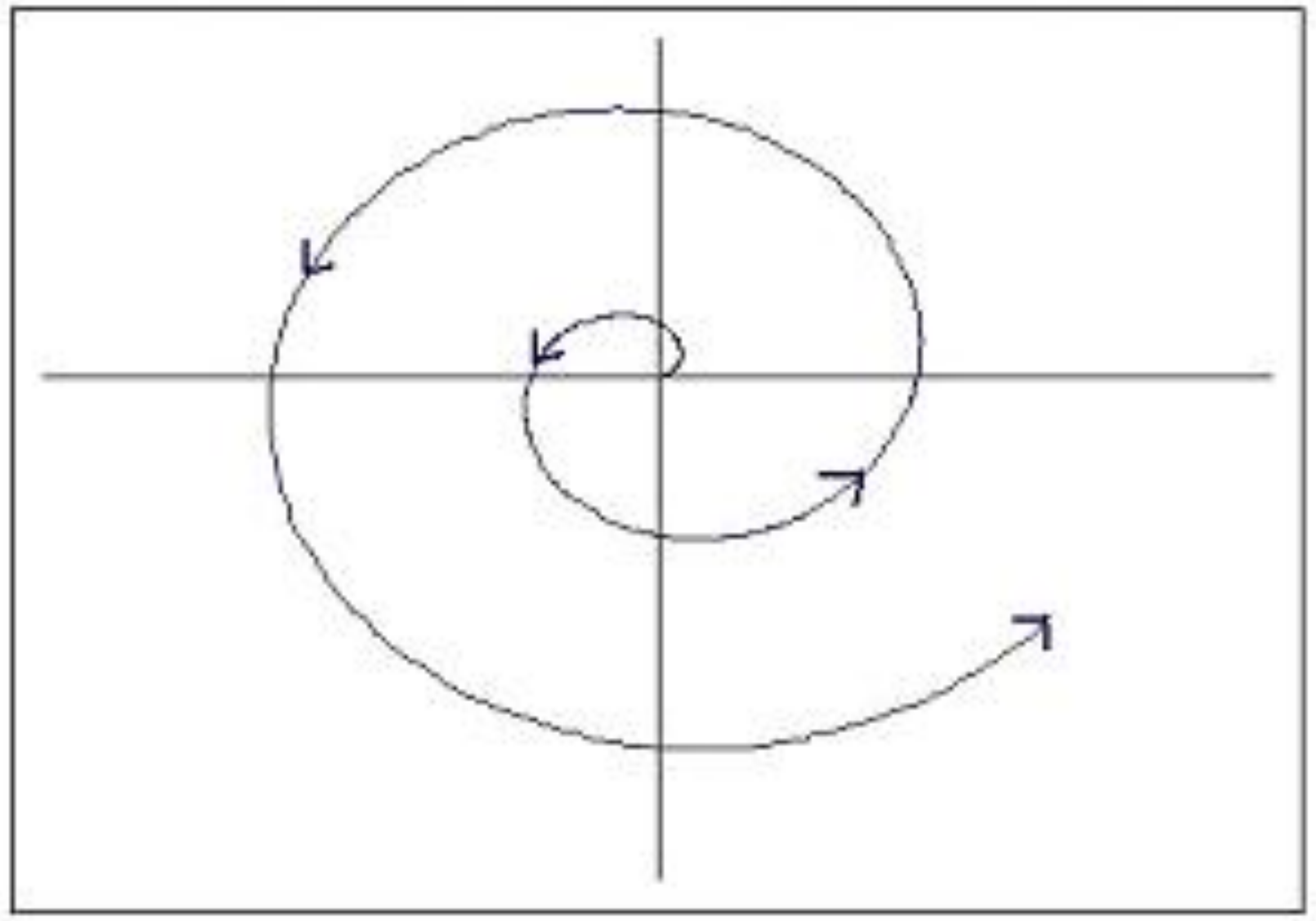


Stability (discrete time)

$$\dot{x} = Ax$$

$$x(k+1) = \tilde{A}x(k)$$



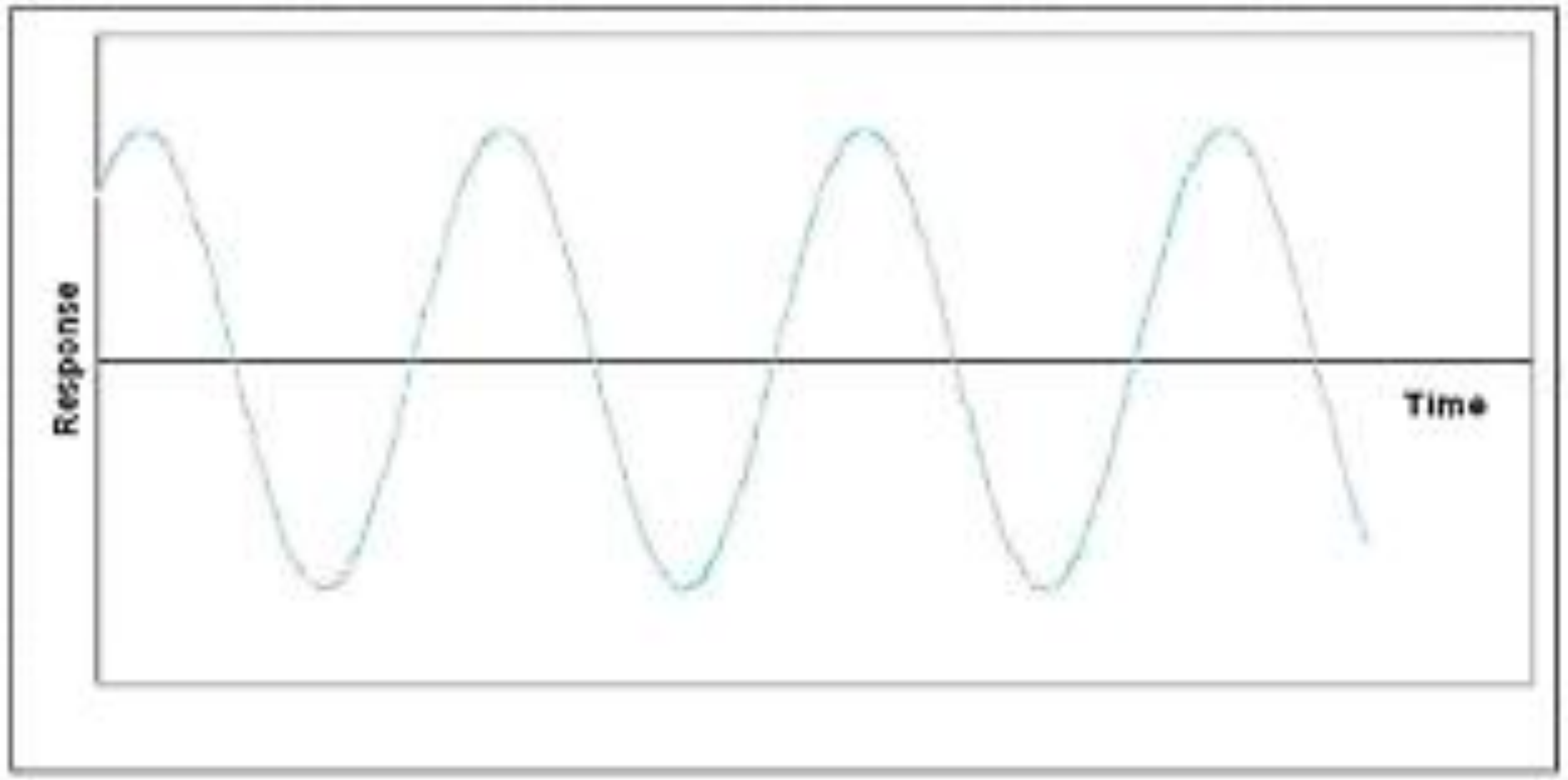


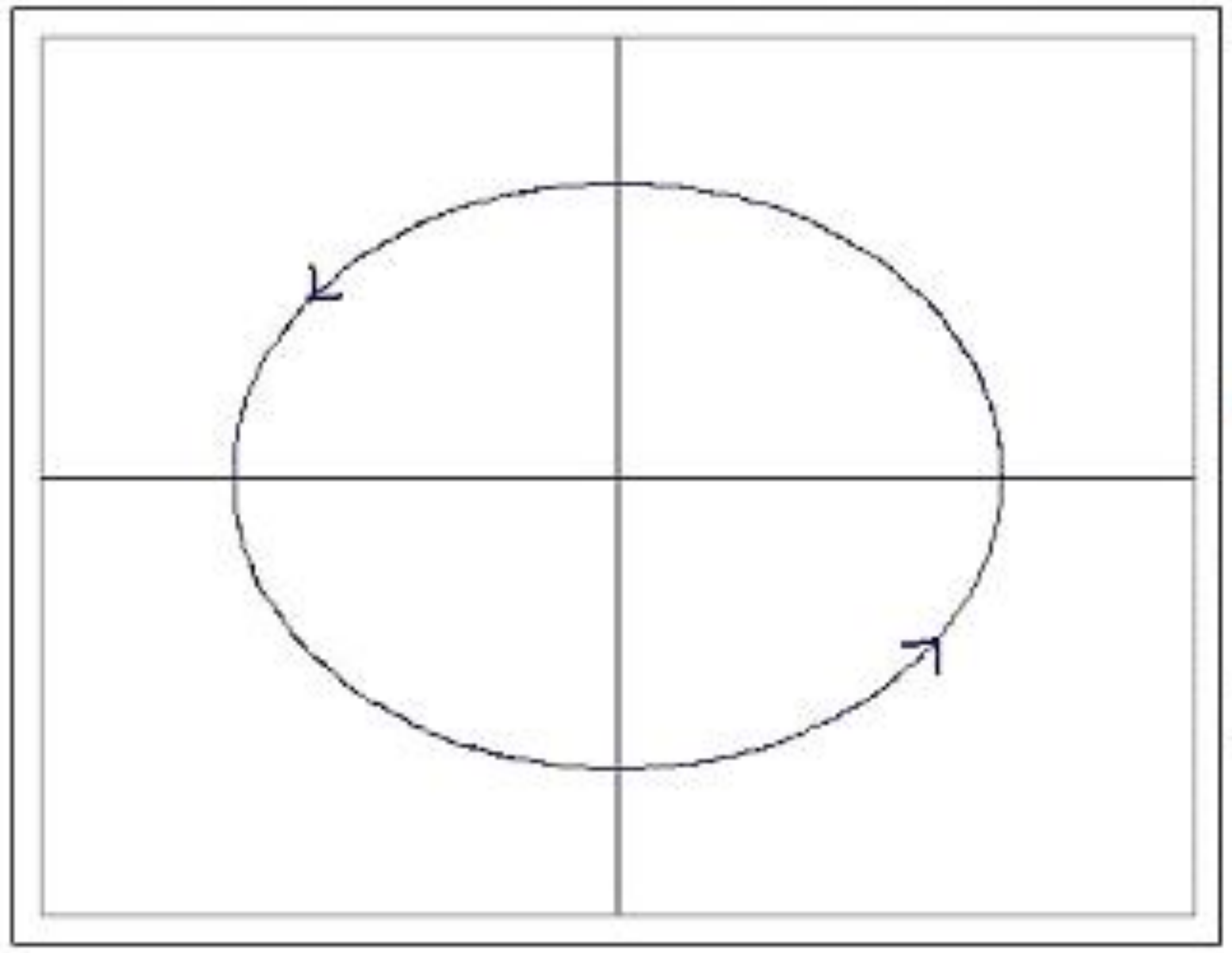
Unstable
(Positive real part, R>1)

Stability (discrete time)

$\dot{x} = Ax$

$x(k+1) = \tilde{A}x(k)$



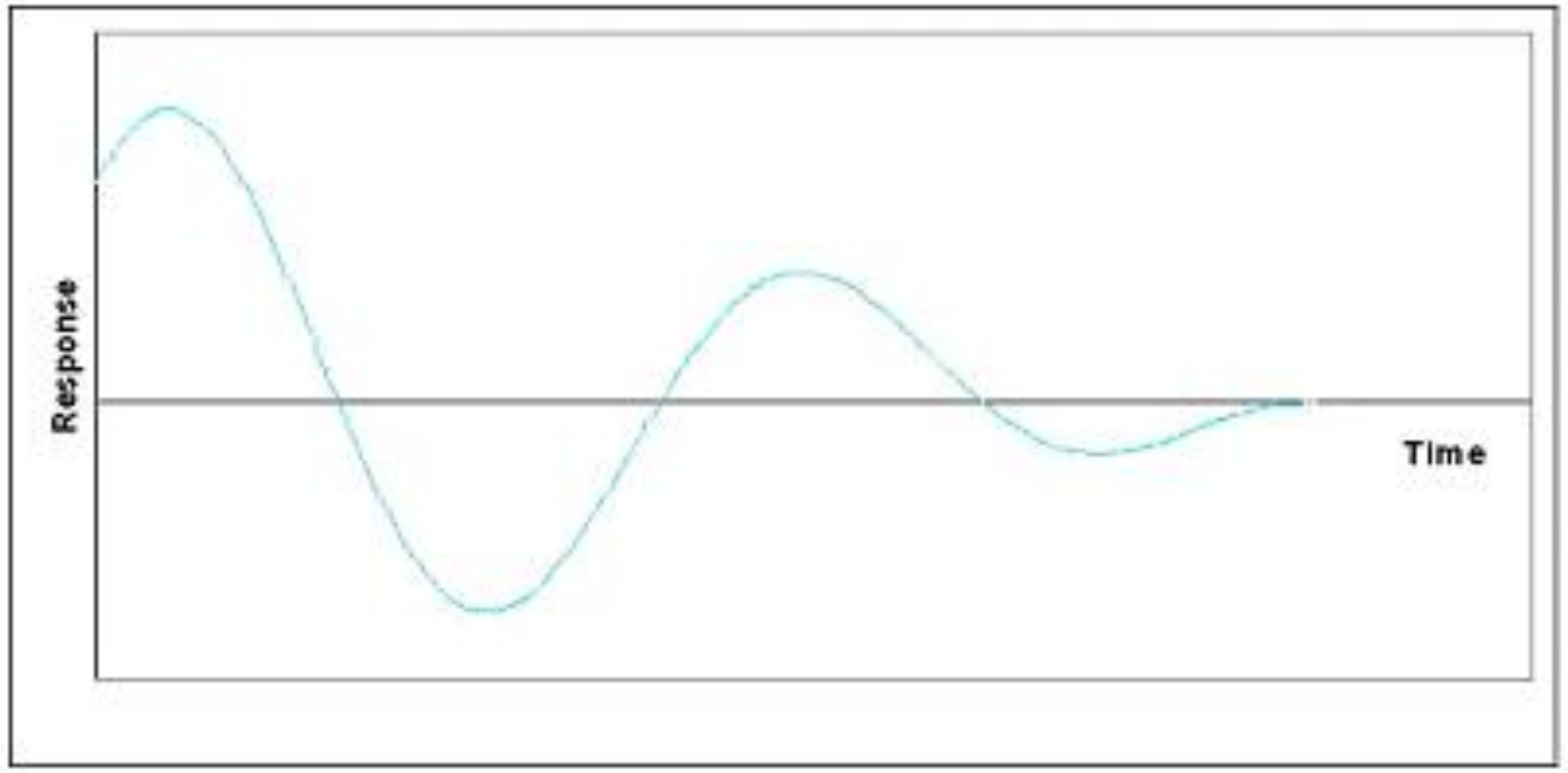


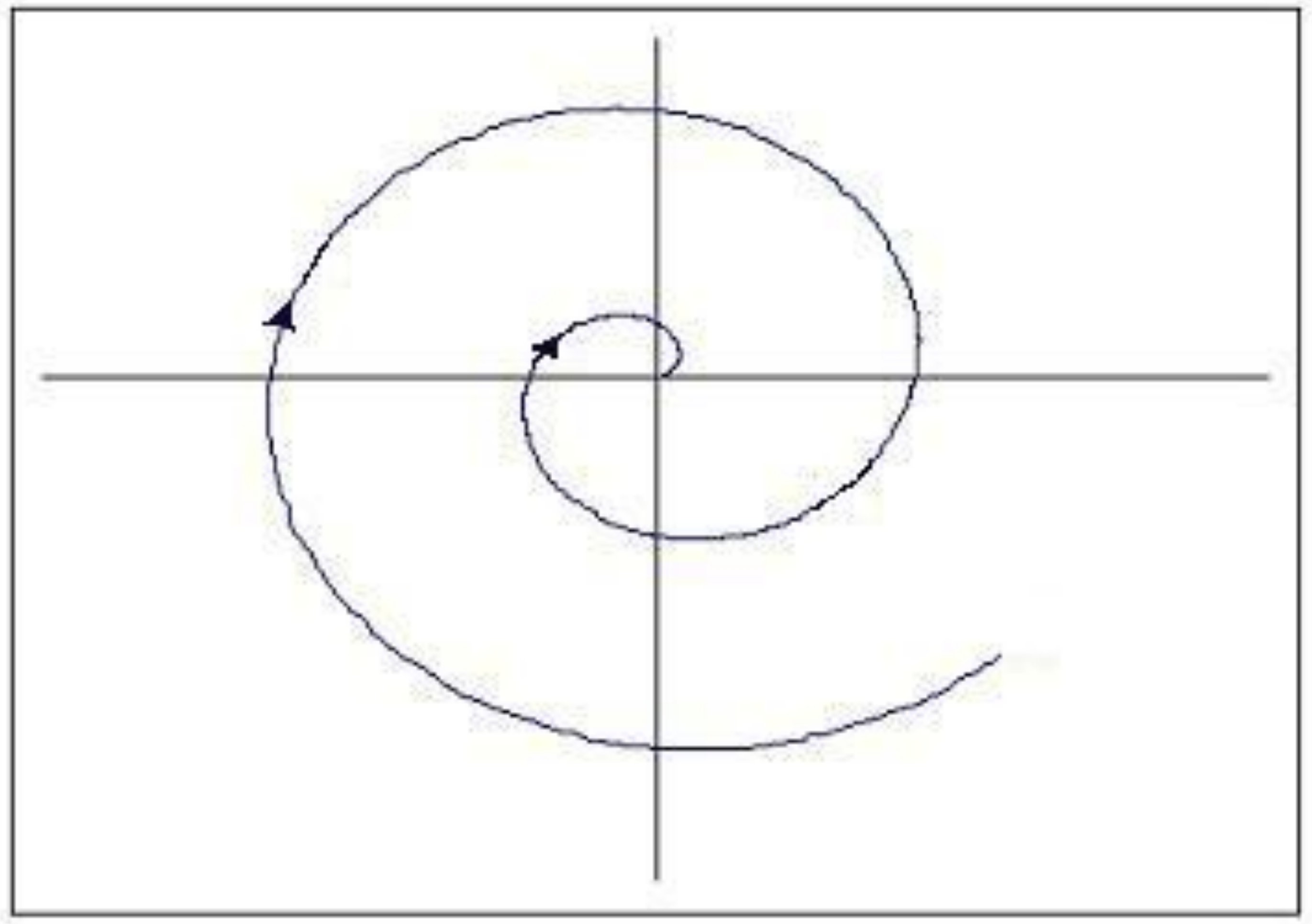
Critically stable
(Zero real part, R = 1)

Stability (discrete time)

$$\dot{x} = Ax$$

$$x(k+1) = \tilde{A}x(k)$$





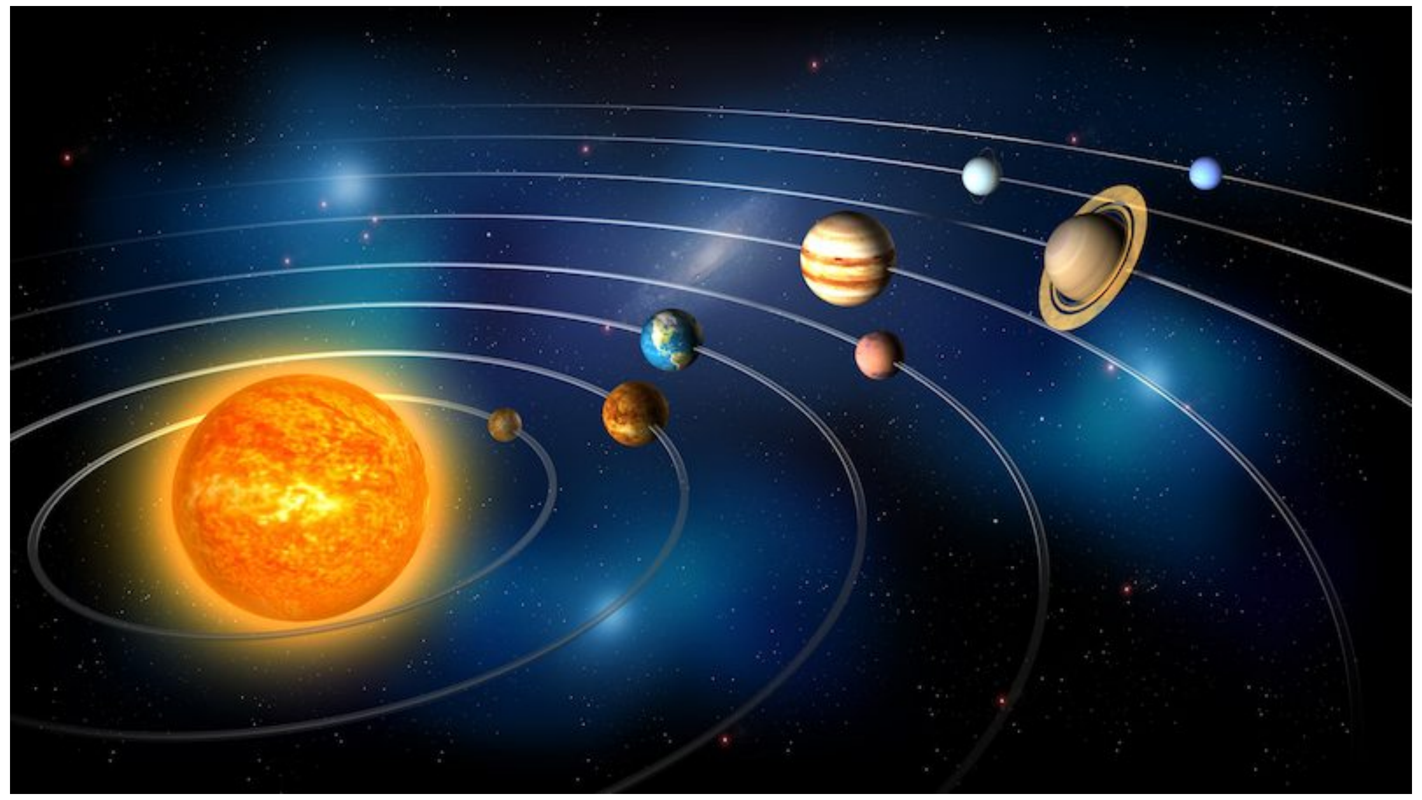
Stable
(Negative real part, R<1)

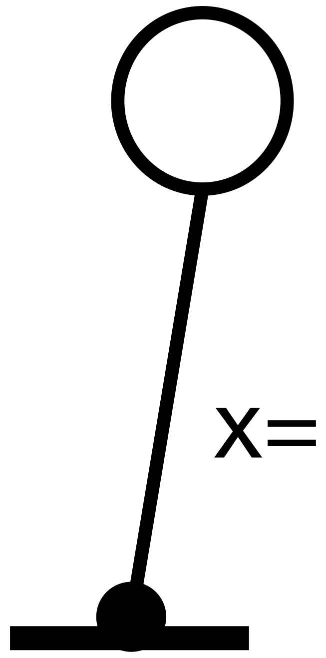
Linearizing Nonlinear Systems

Basic steps to linearize nonlinear systems

$$\dot{x} = f(x) \longrightarrow \dot{x} = Ax$$

- Find some fixed points
  - $\bar{x}$ st $f(\bar{x}) = 0$
- Linearize about them
  - $\left.\frac{Df}{Dx}\right|_{\bar{x}} = \left[\frac{\delta f_i}{\delta x_j}\right]$ "Jacobian"





x= $\begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix}$

Example:

$$\dot{x}_1 = f_1(x_1, x_2) = x_1 x_2$$

$$\dot{x}_2 = f_2(x_1, x_2) = x_1^2 + x_2^2$$

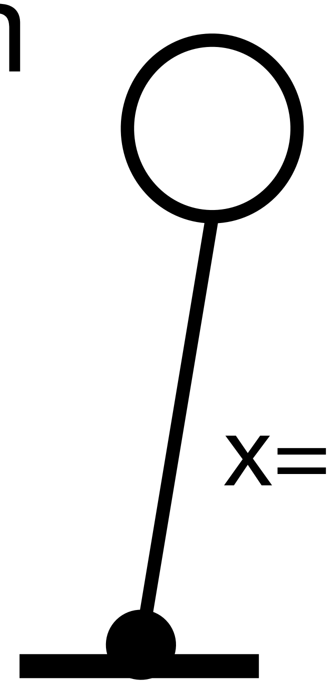
$$\frac{Df}{Dx} = \begin{bmatrix} \frac{\delta f_1}{\delta x_1} & \frac{\delta f_1}{\delta x_2} \\ \frac{\delta f_2}{\delta x_1} & \frac{\delta f_2}{\delta x_2} \end{bmatrix}$$

$$\frac{Df}{Dx} = \begin{bmatrix} x_2 & x_1 \\ 2x_1 & 2x_2 \end{bmatrix}$$ Evaluate at $\bar{x}$

Basic steps to linearize nonlinear systems

$$\dot{x} = f(x) \longrightarrow \dot{x} = Ax$$

- Find some fixed points
  - $\bar{x}$ st $f(\bar{x}) = 0$
- Linearize about them
  - $\left.\dfrac{Df}{Dx}\right|_{\bar{x}} = \left[\dfrac{\delta f_i}{\delta x_j}\right]$
- If you zoom in on $\bar{x}$, your system will look linear!



x= $\begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$

Example:

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$$\dot{x}_2 = f_2(x_1, x_2) = x_1^2 + x_2^2$$

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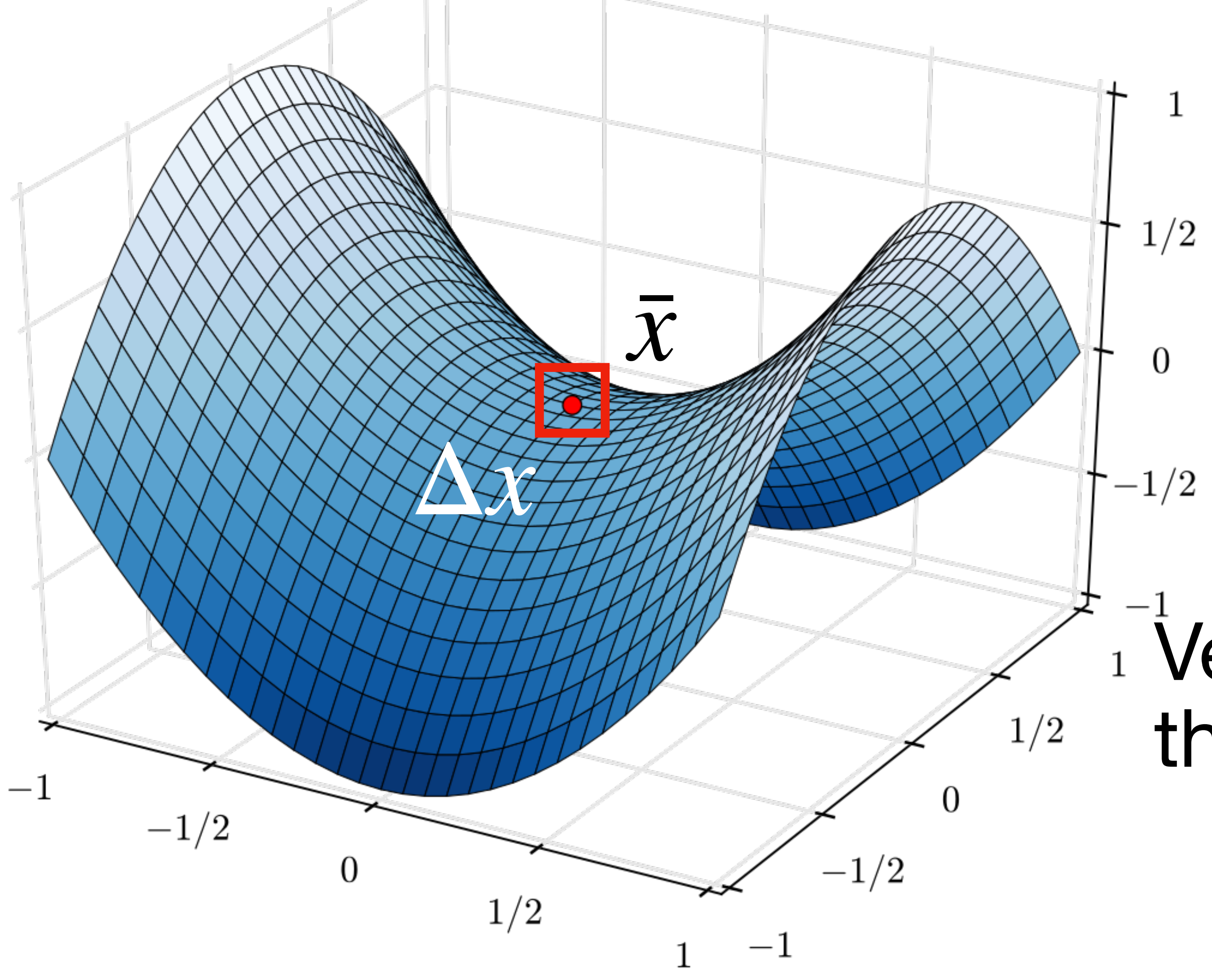

Vector field of the dynamics

$$\dot{x} = f(x)$$

$$\dot{x} = \cancelto{0}{f(\bar{x})} + \left.\frac{Df}{Dx}\right|_{\bar{x}}(x-\bar{x}) + \cancelto{\ll 1}{\left.\frac{D^2 f}{D^2 x}\right|_{\bar{x}}(x-\bar{x})^2 + \left.\frac{D^3 f}{D^3 x}\right|_{\bar{x}}(x-\bar{x})^3 + \ldots}$$

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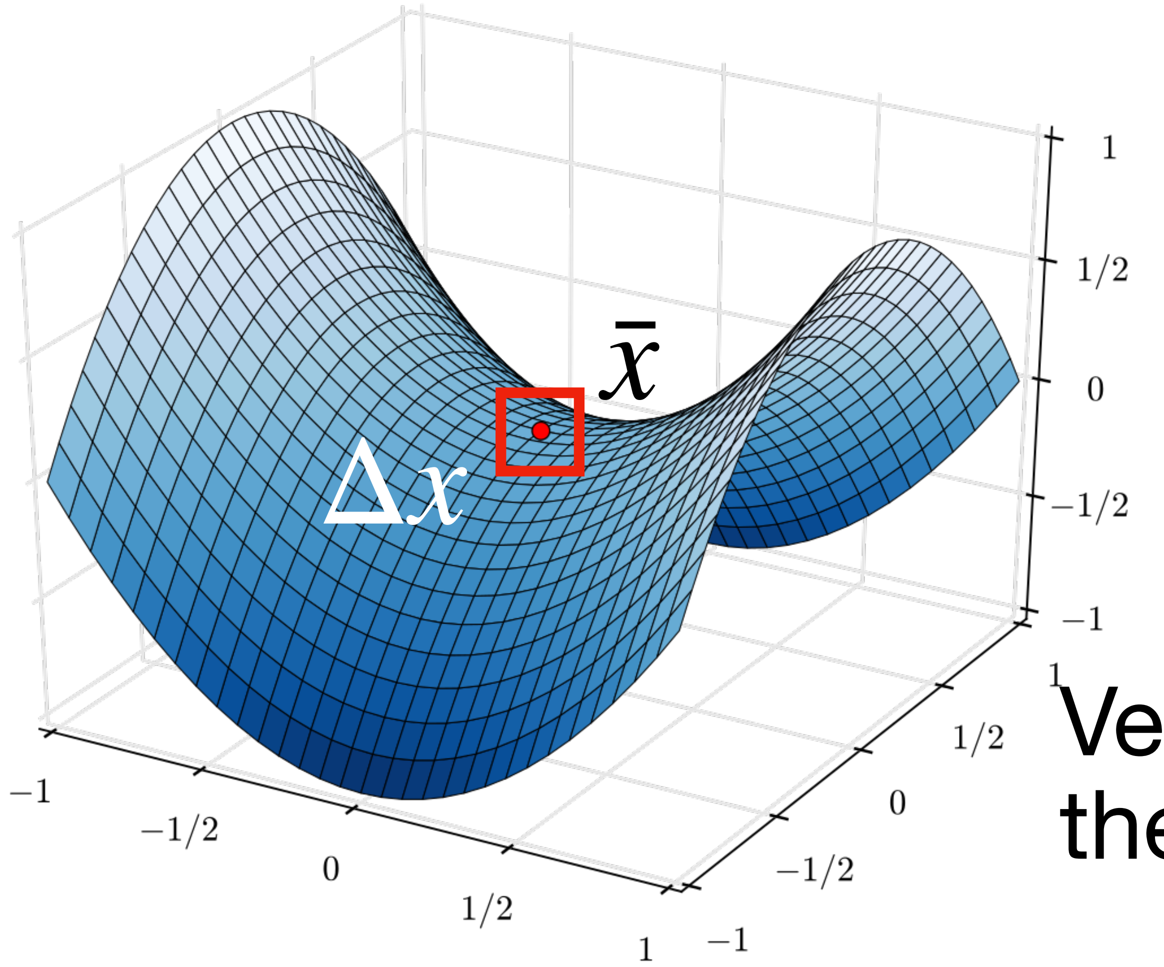
  - $\left.\dfrac{Df}{Dx}\right|_{\bar{x}} = \left[\dfrac{\delta f_i}{\delta x_j}\right]$
- If you zoom in on $\bar{x}$, your system will look linear!
- Good control will keep you near the fixed point, where the model is valid!



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$$\Delta\dot{x} = \left.\frac{Df}{Dx}\right|_{\bar{x}}(\Delta x) \longrightarrow \boxed{\Delta\dot{x} = A\Delta x}$$

Review

- Linear system: $\dot{x} = Ax$
- Solution: $x(t) = e^{At}x(0)$
- Eigenvectors: $T = \begin{bmatrix} \xi_1 & \xi_2 & \dots & \xi_n \end{bmatrix}$
- Eigenvalues: $D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$
- Linear Transform: $AT = TD$
- Solution: $e^{At} = e^{TDT^{-1}t}$
- Mapping from x to z to x: $x(t) = Te^{Dt}T^{-1}x(0)$
- Stability in continuous time: $\lambda = a + ib$, stable iff $a < 0$
- Discrete time: $x(k+1) = \tilde{A}x(k)$, where $\tilde{A} = e^{A\Delta t}$
- Stability in discrete time: $\tilde{\lambda}^n = R^n e^{in\theta}$, stable iff $R < 1$
- Nonlinear systems: $\dot{x} = f(x)$
- Linearization: $\left.\frac{Df}{Dx}\right|_{\bar{x}}$