Particle Filters and SLAM

Fast Robots, ECE4160/5160, MAE 4190/5190

E. Farrell Helbling, 4/15/25

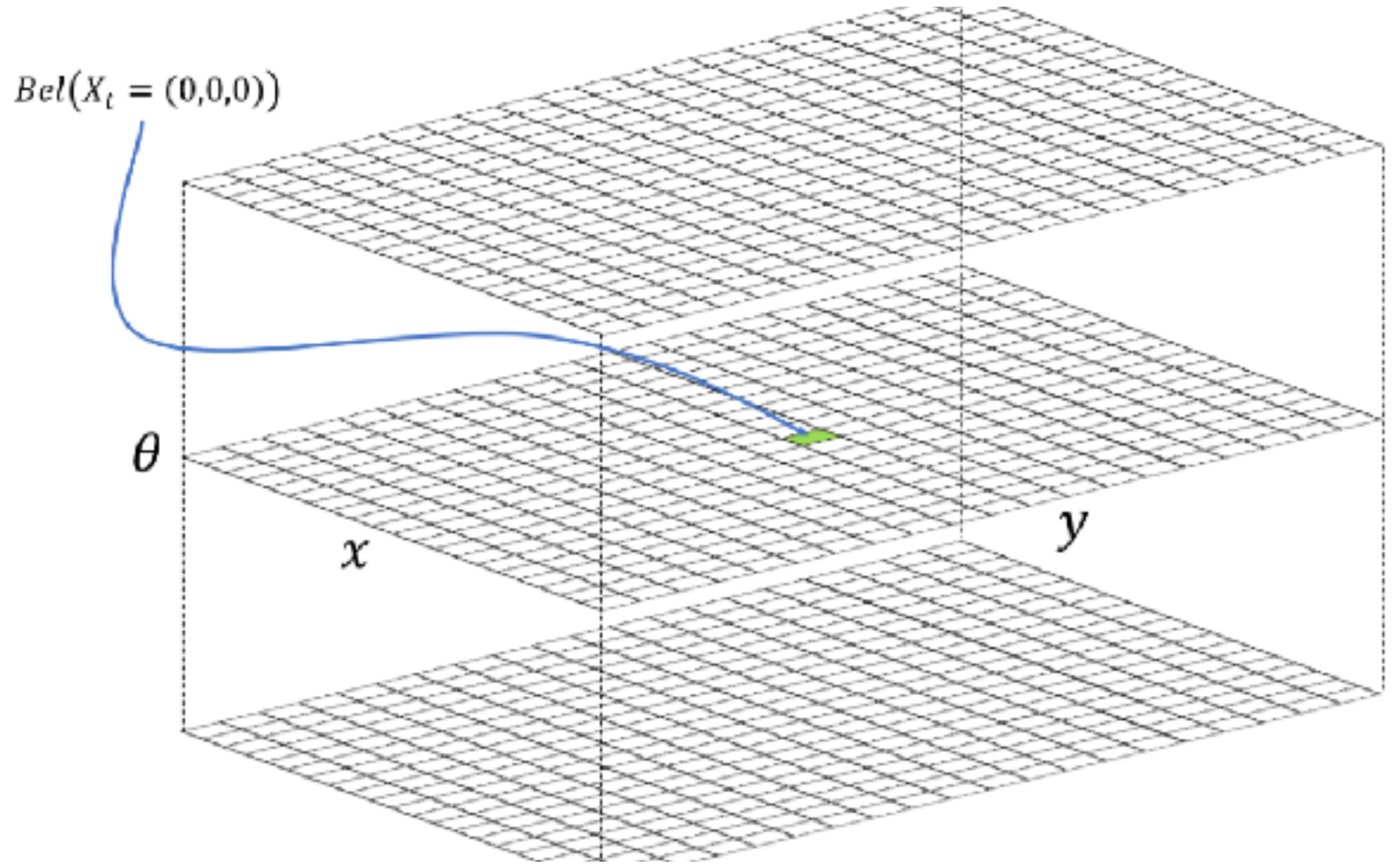
Class Action Items

- Lab 10: localization (sim) starts today!
  - The lab is graded S/U (all of the code is already posted on previous websites)
  - If you do get an unsatisfactory, you will need to redo it.
- About to send a bunch of google forms your way
  - ECE Robotics Day availability (if you are unavailable it’s because of another course conflict)
  - Lab 8 votes on stunts and bloopers
- This is the last technical content lecture of the course!

Lab 10 Localization

- This lab is graded S/U
- Tasks
  - Read the full lab and the notebook (before you show up to lab)
  - Perform grid localization for the sample trajectory
  - Video demo
  - Discuss:
    - Control
    - Motion model and the prediction step
    - Sensor model and the update step
    - Choosing parameters, effect of changing parameters
    - Ways to mitigate computational load
    - Evaluate the Bayes Filter
    - Evaluate how well this will work on your robot

Grid-based localization



- Simple
- … but it is computationally expensive for large workspaces

Algorithm Bayes_Filter $(bel(x_{t-1}), u_t, z_t)$:

  for all x_t do

    $\overline{bel}(x_t) = \Sigma_{x_{t-1}}\ p(x_t \mid u_t, x_{t-1})\ bel(x_{t-1})$

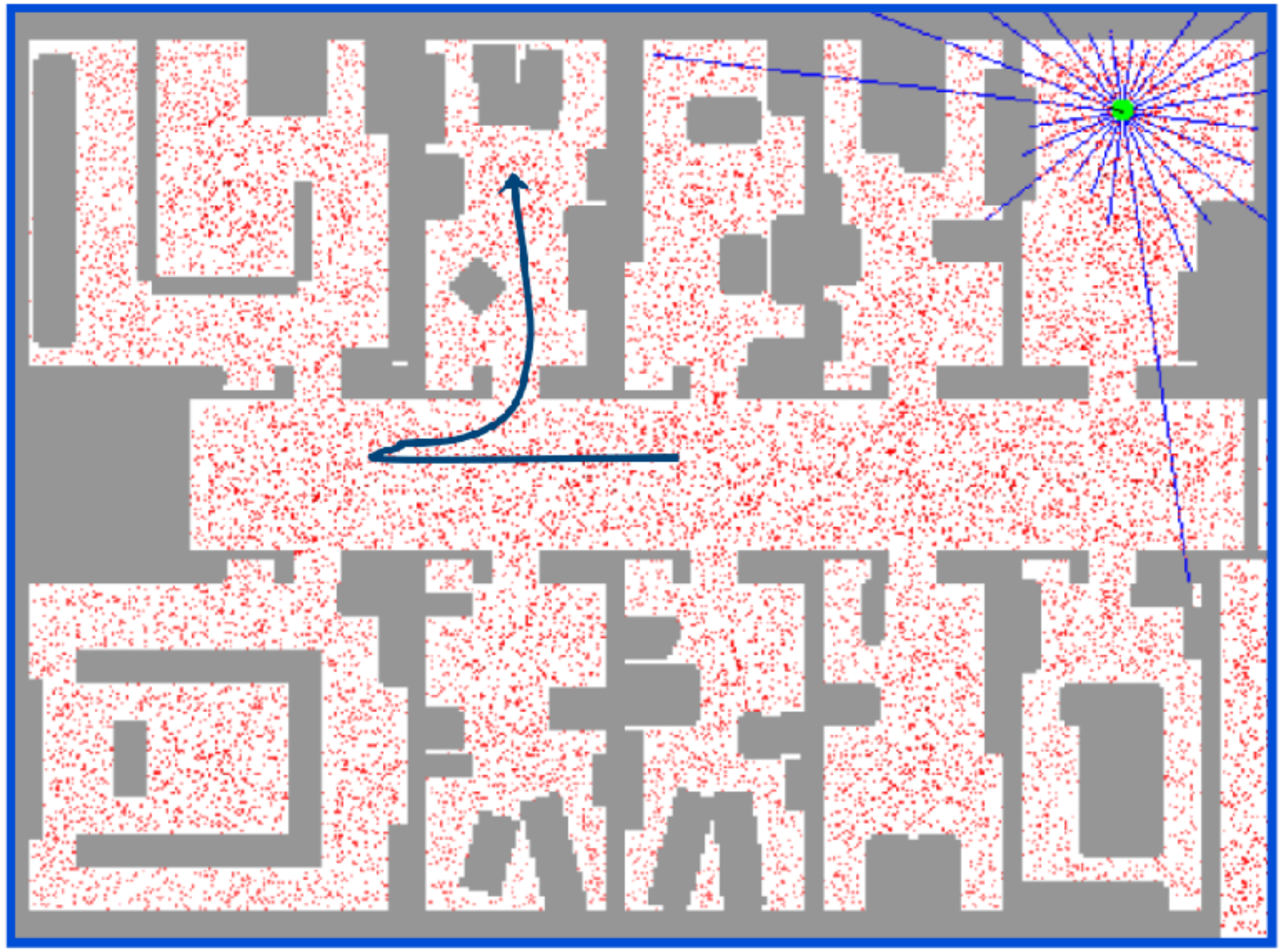
    $bel(x_t) = \eta\ p(z_t \mid x_t)\ \overline{bel}(x_t)$

  end **for**

return $bel(x_t)$

Monte Carlo Localization

- Non-parametric approach based on Particle Filters
- Model the distribution by samples
  - Prediction step
    - Draw from the samples
    - Move forward based on motion model
  - Update step
    - Weigh samples by their importance
    - Sensor model
  - Resample based on their weight
- The more samples we use, the better the estimate!



Monte Carlo Localization

- Non-parametric approach based on Particle Filters
- Model the distribution by samples

1. **Algorithm Bayes_Filter** $(bel(x_{t-1}), u_t, z_t)$:
2. **for** all x_t do
3. $\overline{bel}(x_t) = \Sigma_{x_{t-1}} \; p(x_t \mid u_t, x_{t-1}) \; bel(x_{t-1})$
4. $bel(x_t) = \eta \; p(z_t \mid x_t) \; \overline{bel}(x_t)$
5. end **for**
6. **return** $bel(x_t)$

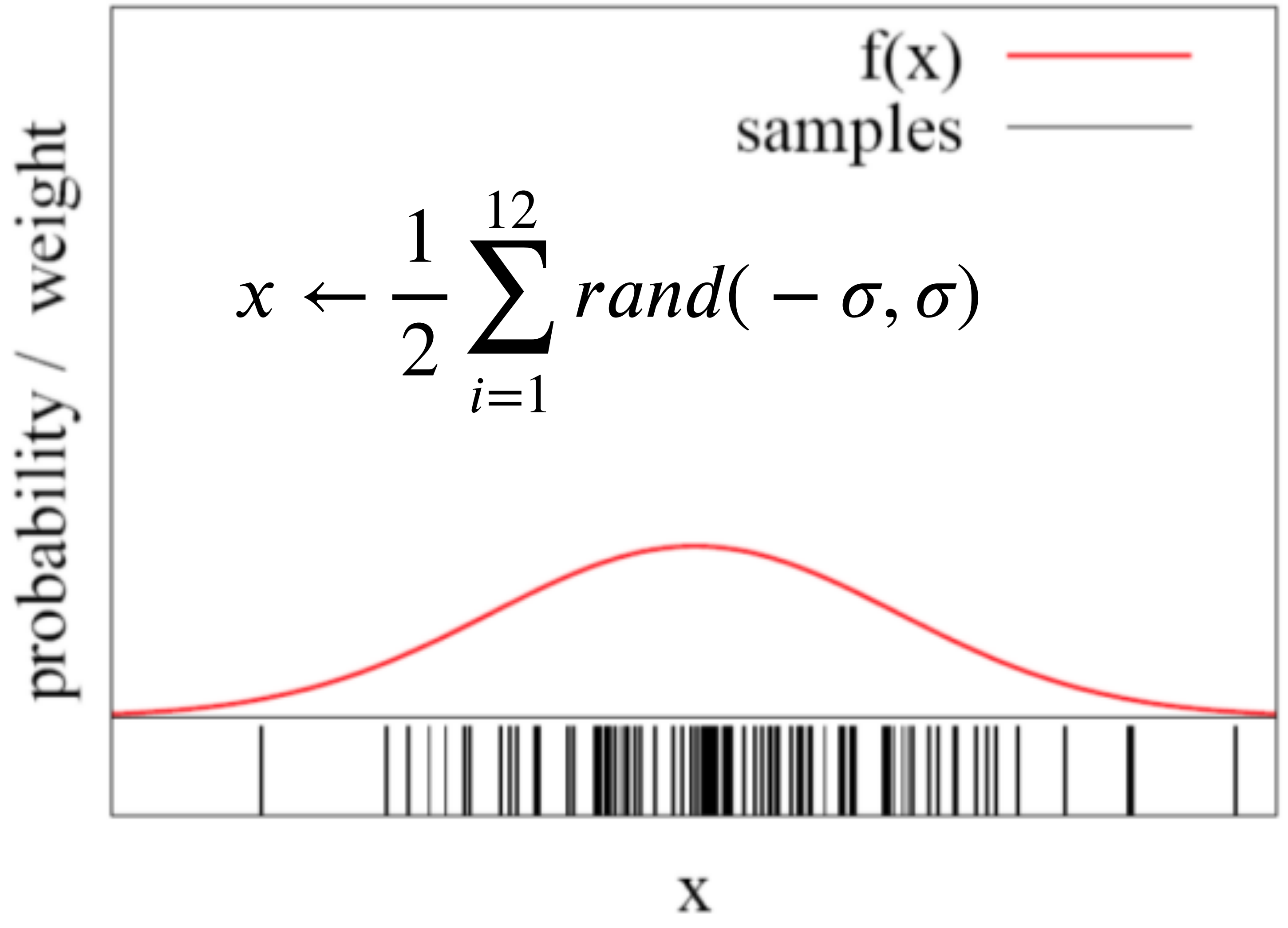
Draw x_t^i from $p(x_t \mid u_t, x_{t-1}^i)$

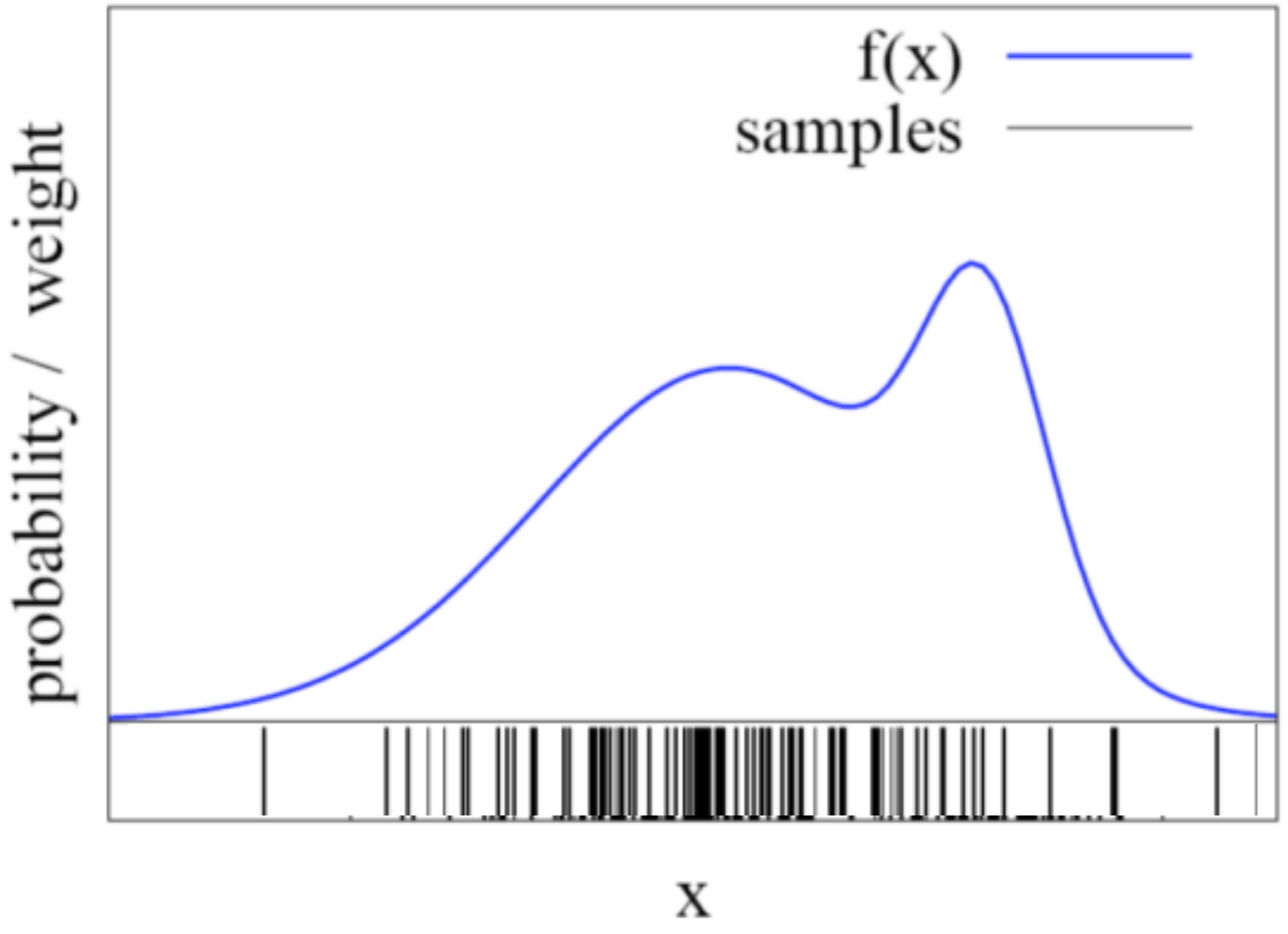
Prior samples

Importance factor $w_t^i \propto p(z_t \mid x_t)$

Monte Carlo Localization

- How do you obtain samples from an arbitrary distribution?
  - Closed form solution for a uniform distribution
  - Closed form solution for a Gaussian distribution

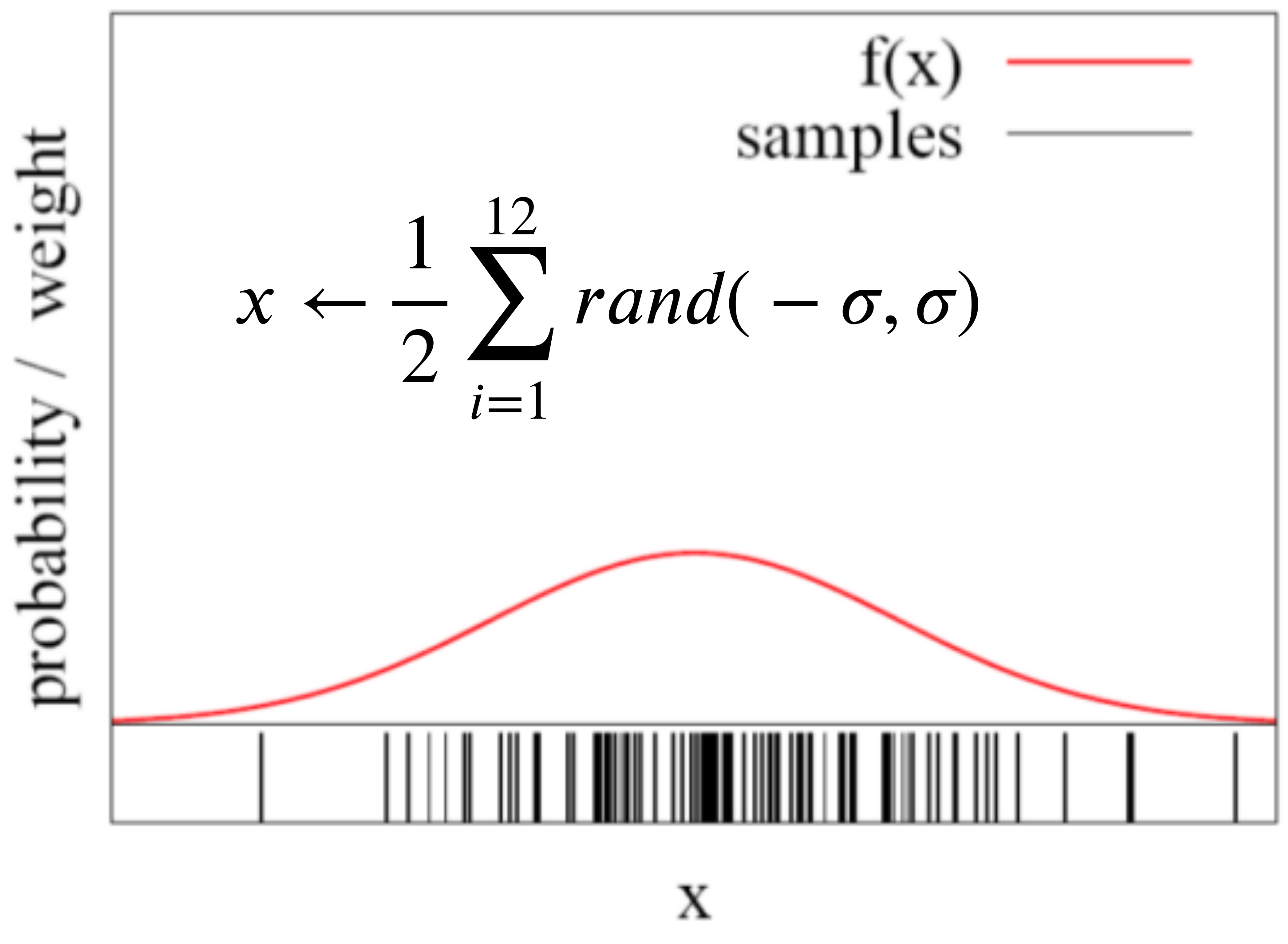


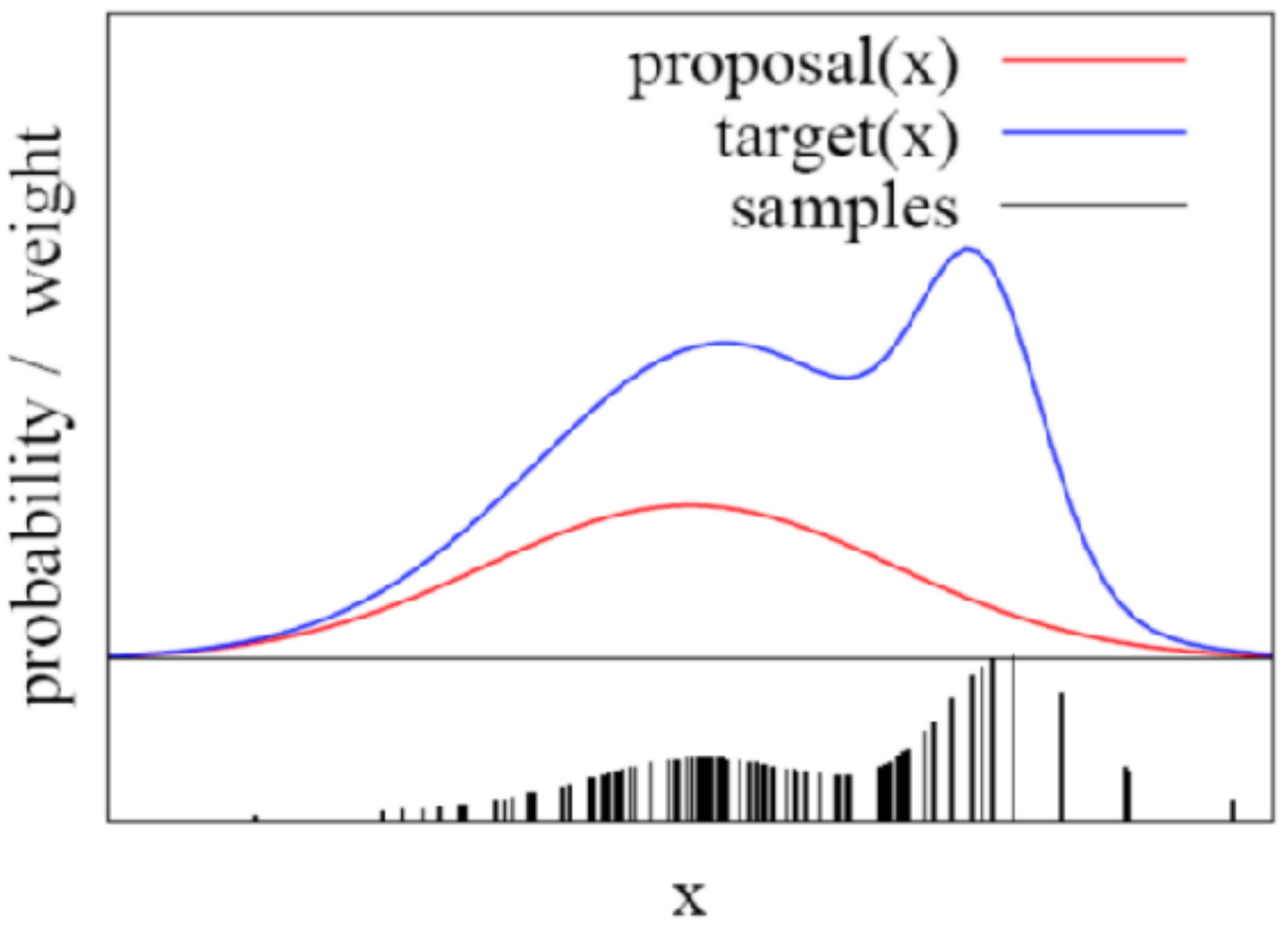


Fast Robots 2025

Monte Carlo Localization

- How do you obtain samples from an arbitrary distribution?
  - Closed form solution for a uniform distribution
  - Closed form solution for a Gaussian distribution
- Use a proposal distribution to generate samples from the target distribution
- Account for differences using a weight
  - w = target/ proposal





Monte Carlo Localization

- Each particle, *j*, is a pose hypothesis
- Proposal distribution from the motion model
  - $x_t^{[j]} \sim p(x_t \,|\, x_{t-1}, u_t)$
- Correction via the observation model
  - $w_t^{[j]} = \dfrac{target(x_t^{[j]})}{proposal(x_t^{[j]})} = p(z_t \,|\, x_t)$
- Resample
  - Draw sample i with probably $w_t^{[j]}$ and repeat J times

Particle_filter$(\mathcal{X}_{t-1}, u_t, z_t)$**:**

$\quad \bar{\mathcal{X}}_t = \mathcal{X}_t = \emptyset$

$\quad$ for $j = 1$ to J do

$\quad\quad$ sample $x_t^{[j]} \sim p(x_t \mid u_t, x_{t-1}^{[j]})$

$\quad\quad w_t^{[j]} = p(z_t \mid x_t^{[j]})$

$\quad\quad \bar{\mathcal{X}}_t = \bar{\mathcal{X}}_t + \langle x_t^{[j]}, w_t^{[j]} \rangle$

$\quad$ endfor

$\quad$ for $j = 1$ to J do

$\quad\quad$ draw $i \in 1, \ldots, J$ with probability $\propto w_t^{[i]}$

$\quad\quad$ add $x_t^{[i]}$ to $\mathcal{X}_t$

$\quad$ endfor

$\quad$ return $\mathcal{X}_t$

https://www.cs.uml.edu/~holly/teaching/4510and5490/fall2018/Lecture-Particle-Filters.pdf

Monte Carlo Localization

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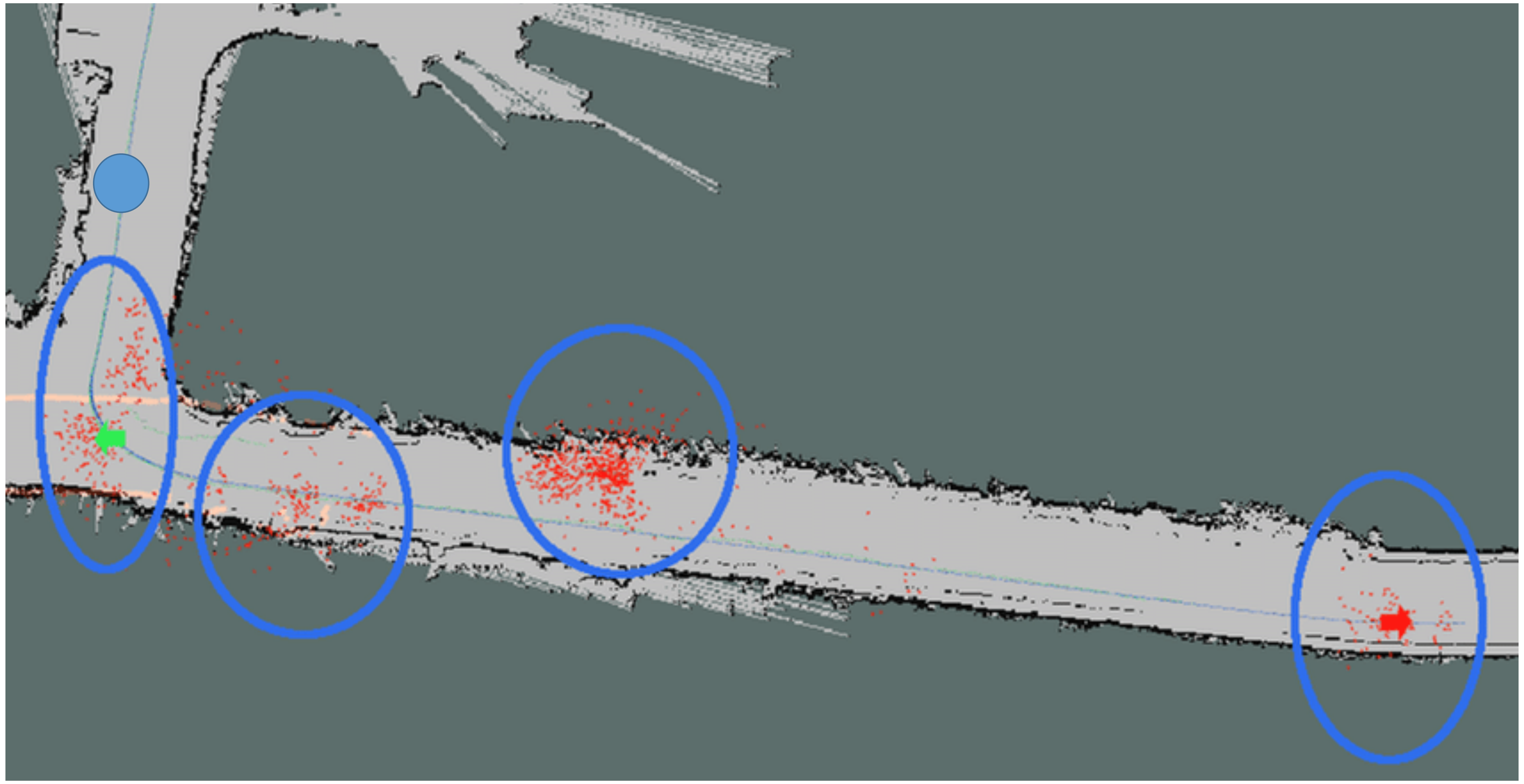
Monte Carlo Localization

- How would you deal with a kidnapped robot?
  - Randomly insert samples proportional to the average likelihood of the particles



- Pros
  - Works well for high-uncertainty scenarios
  - Much more efficient than grid cells
- Cons
  - Scales poorly with higher dimensional workspaces

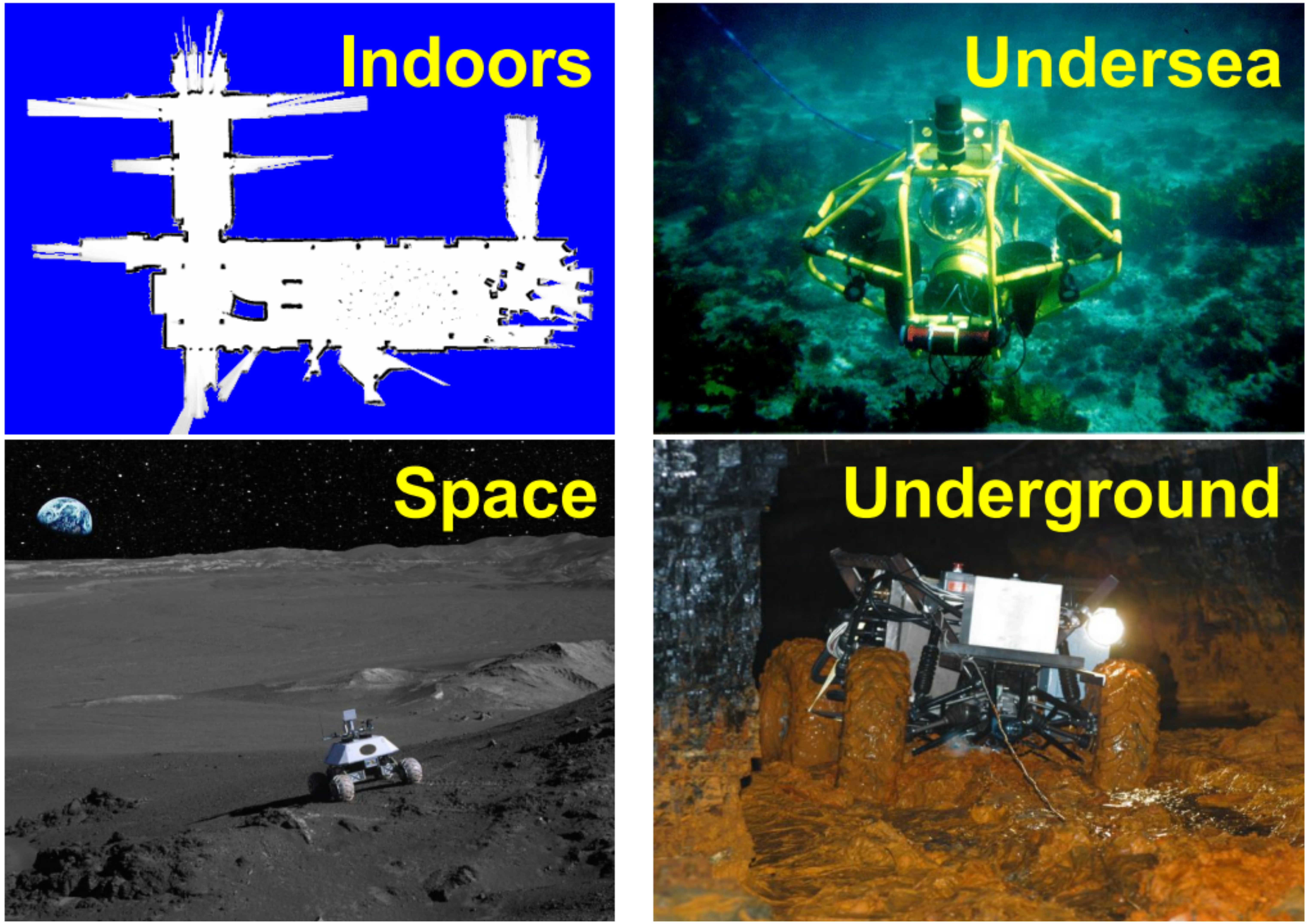
Brief intro to SLAM

Related terms

- State estimation
- Localization **Inferring a location given a map**
- Mapping **Inferring a map given a location**
- SLAM **Learning a map and locating the robot simultaneously**
- Navigation
- Motion planning

Related terms

- State estimation
- Localization
- Mapping
- **SLAM**
- Navigation
- Motion planning



Given all we have learned…

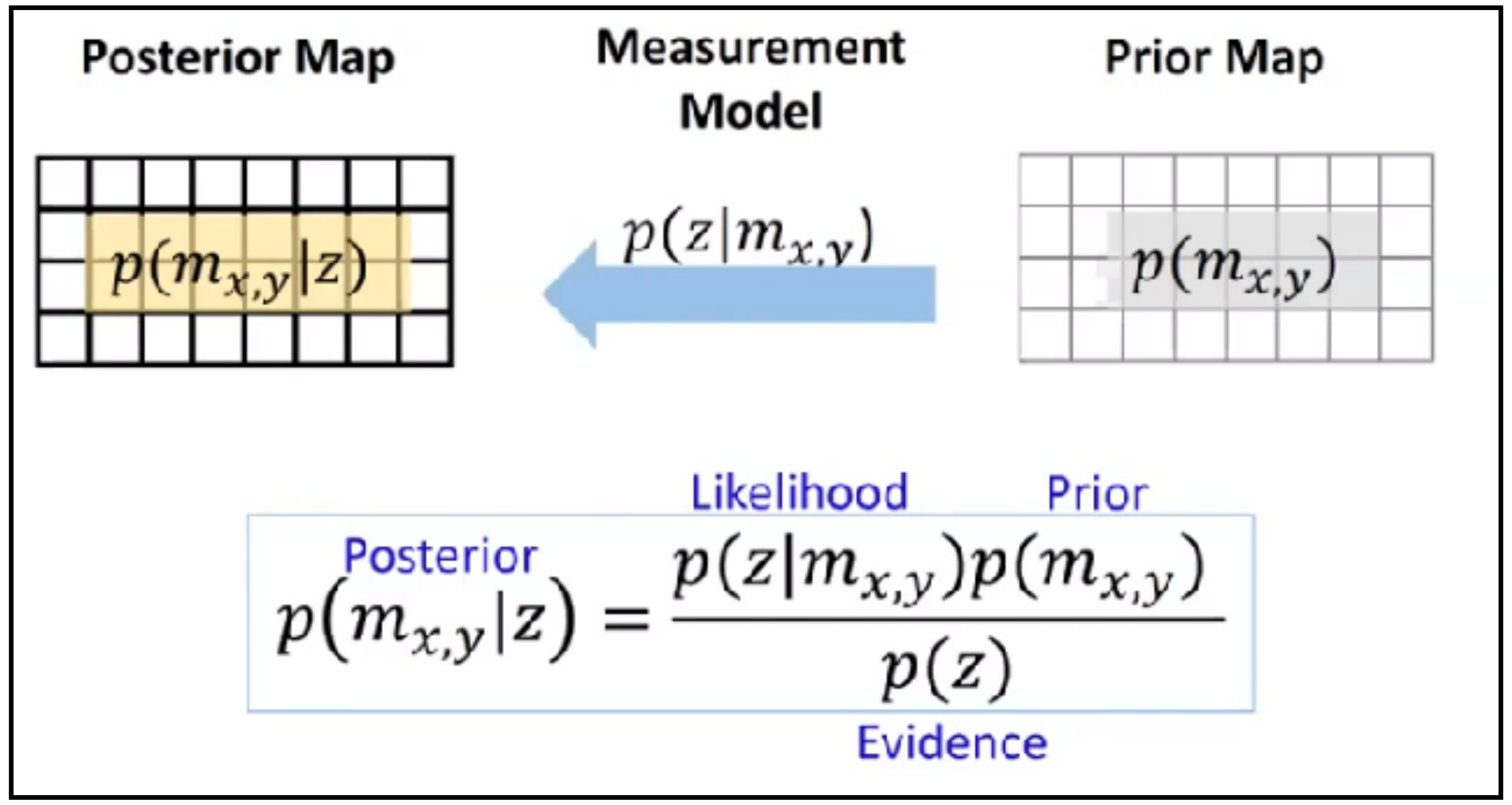
- Transformation matrices
- Sensors and actuators (and probabilistic models)
- Controllers (PID, LQR)
- Observers (KF) Include the map into the state
- Mapping
- Localization
    - Bayes Filter and grid-localization Add grid-occupancy
    - Particle Filter Let particles represent both pose and map
- Graph Search and Planning

… how would you implement SLAM?

(where could your estimate of the map fit in?)

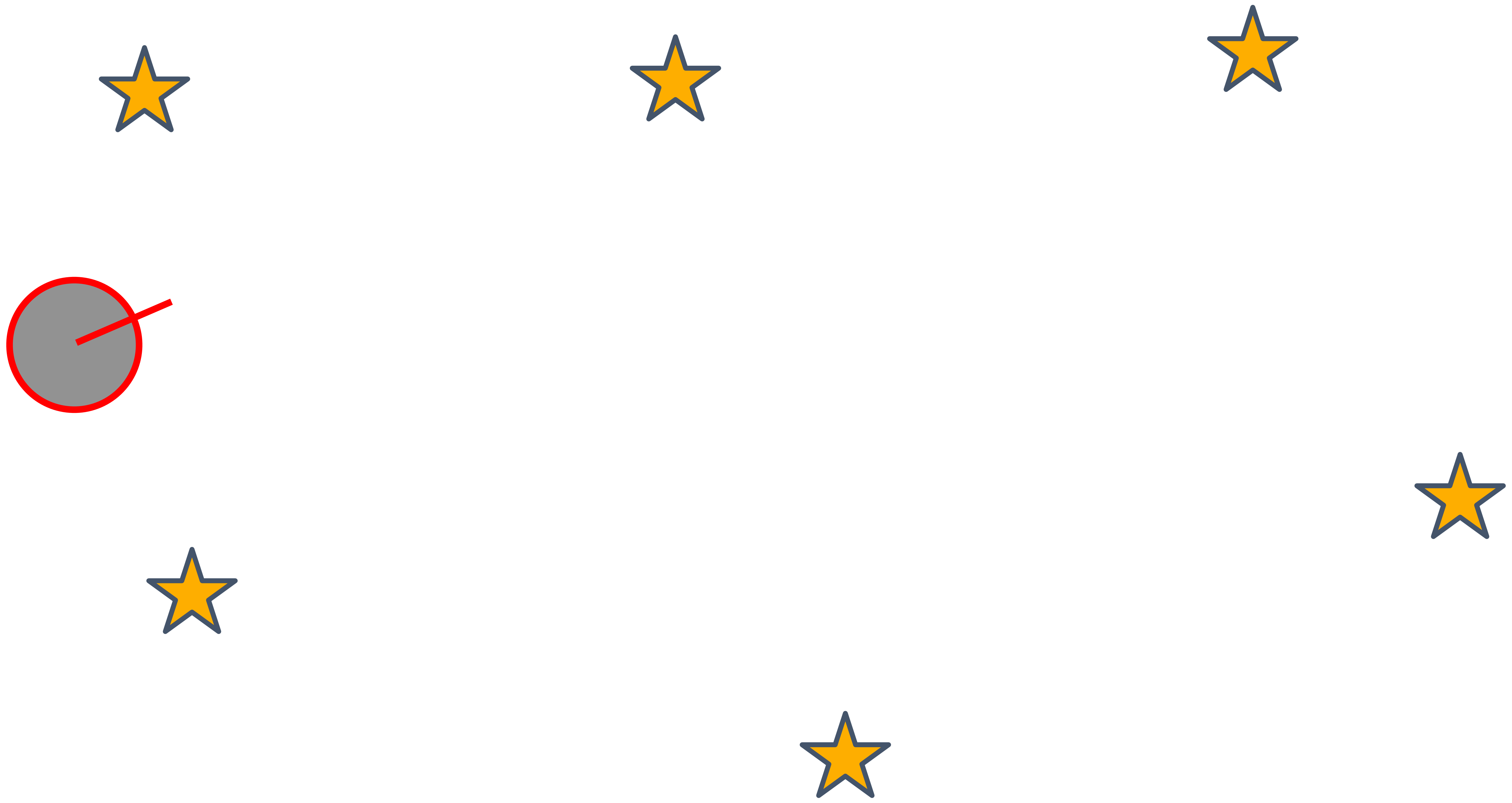
Given all we have learned…

- Markov localization in a grid
  - Localization: estimate your cell pose within the map
  - Mapping: estimate if cells are occupied or not
    - Every grid cell is a random variable
  - SLAM: estimate pose *and* if cells are occupied or not
    - 100x100 grid cells (pretty small map)
    - Localization: (x, y, theta) = 100x100x100 states
    - Map: (x,y) = 10,000 staes
    - SLAM 100x100x100x10,000 states
- Same issue for particle filters…
  - Balance parametric and non-parametric approaches



Why is SLAM hard?

- Robot pose/path and map are both unknown (not independent)
- Map and pose estimates are correlated



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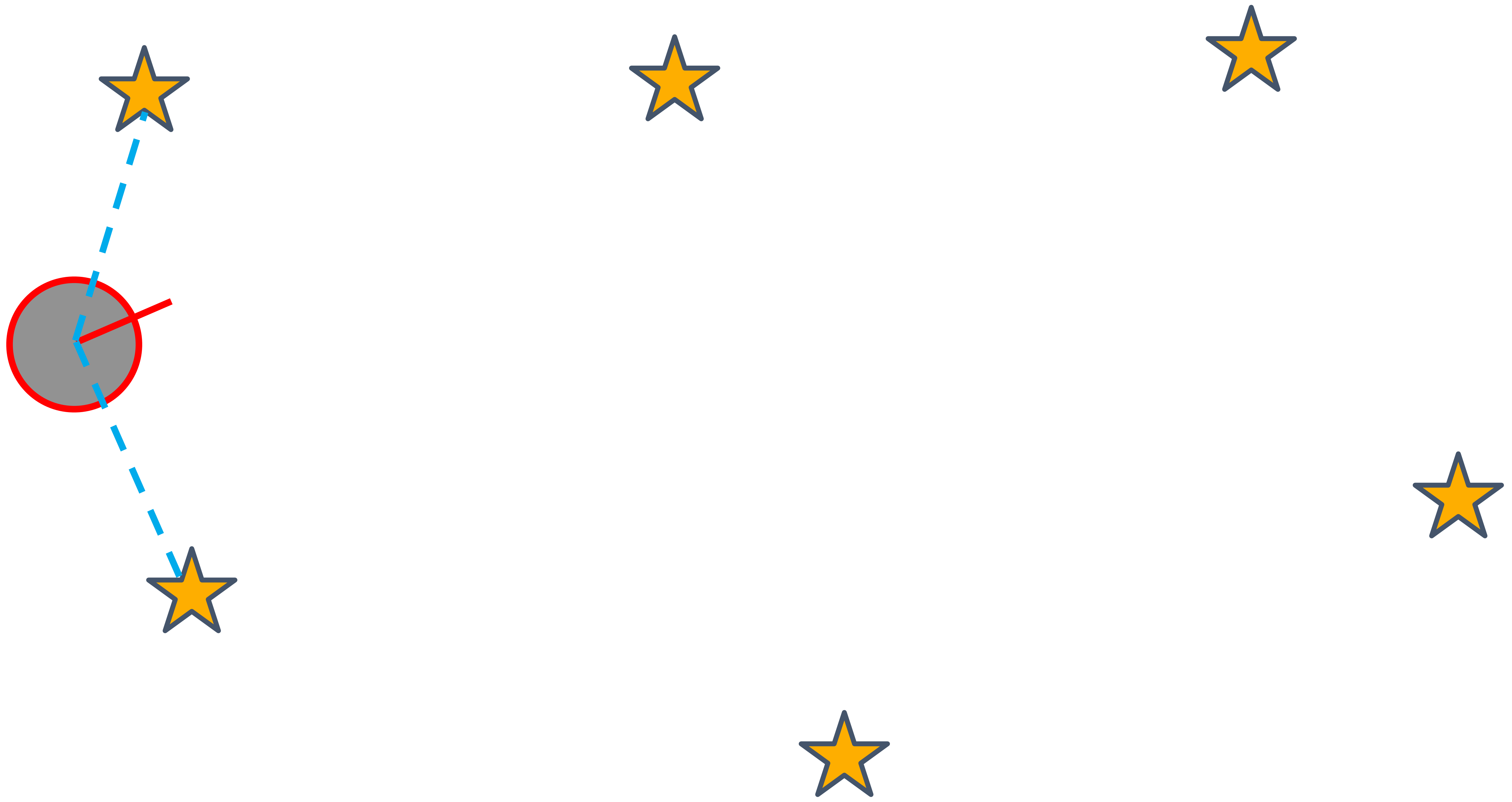

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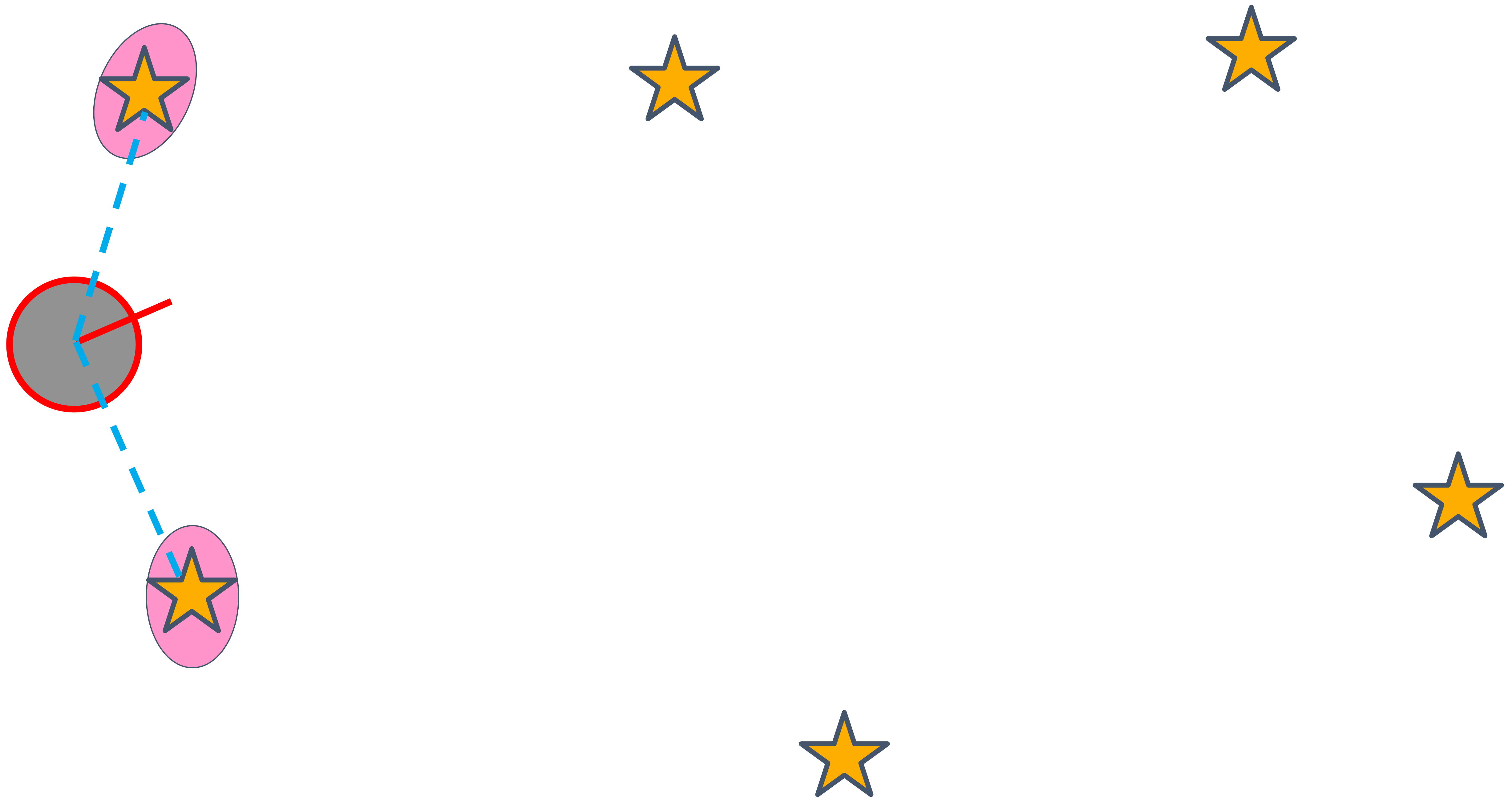

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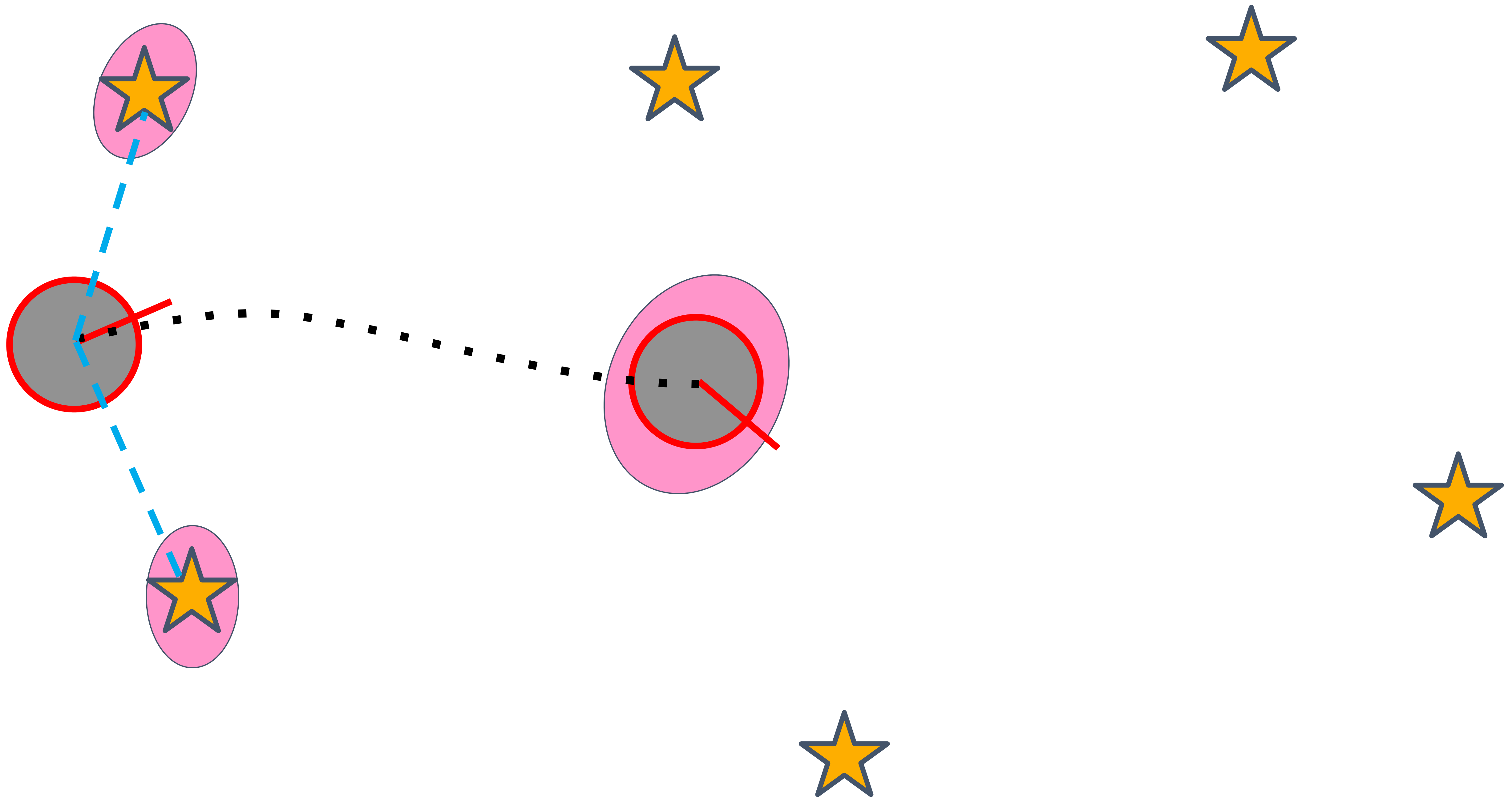

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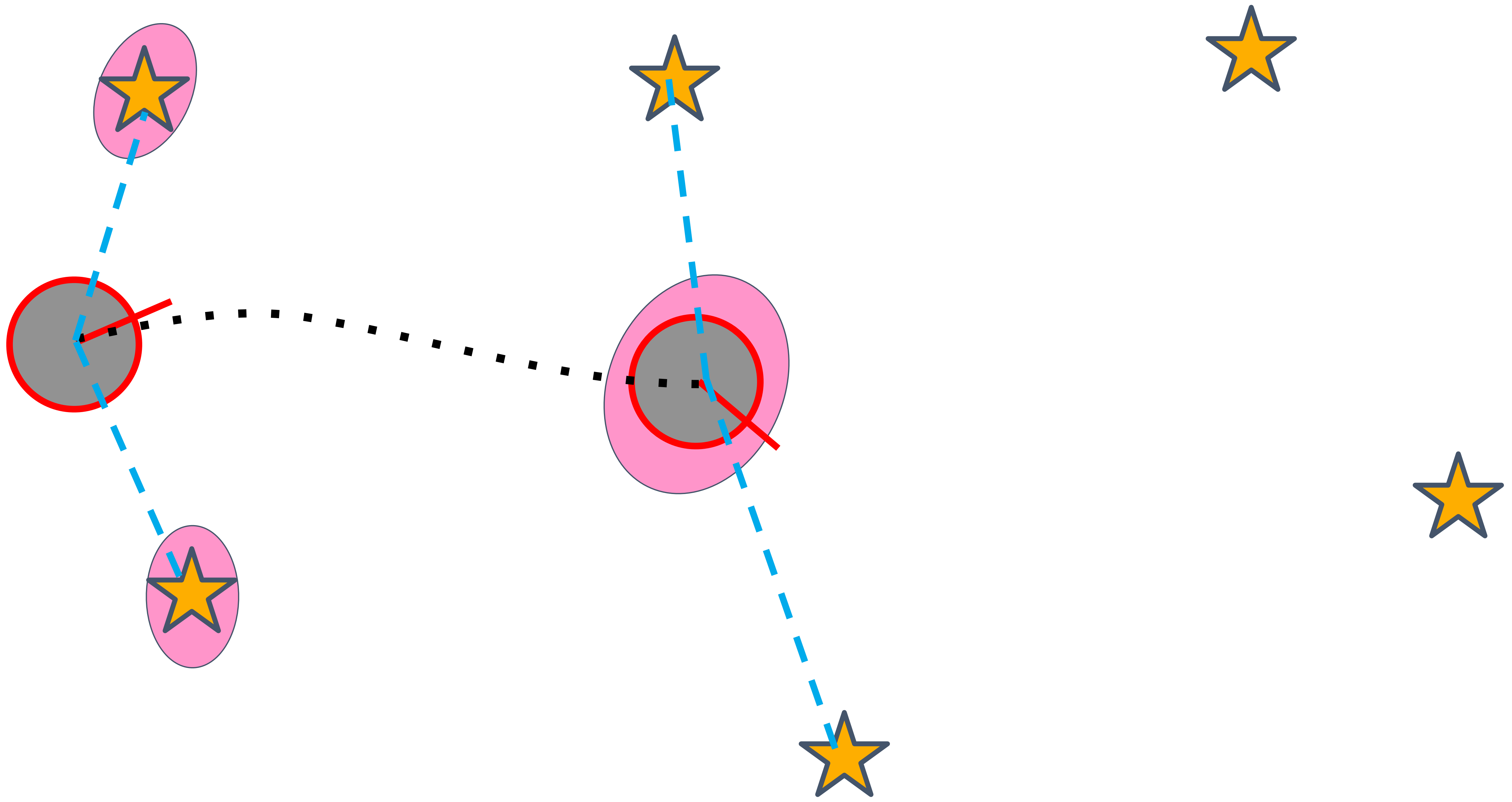

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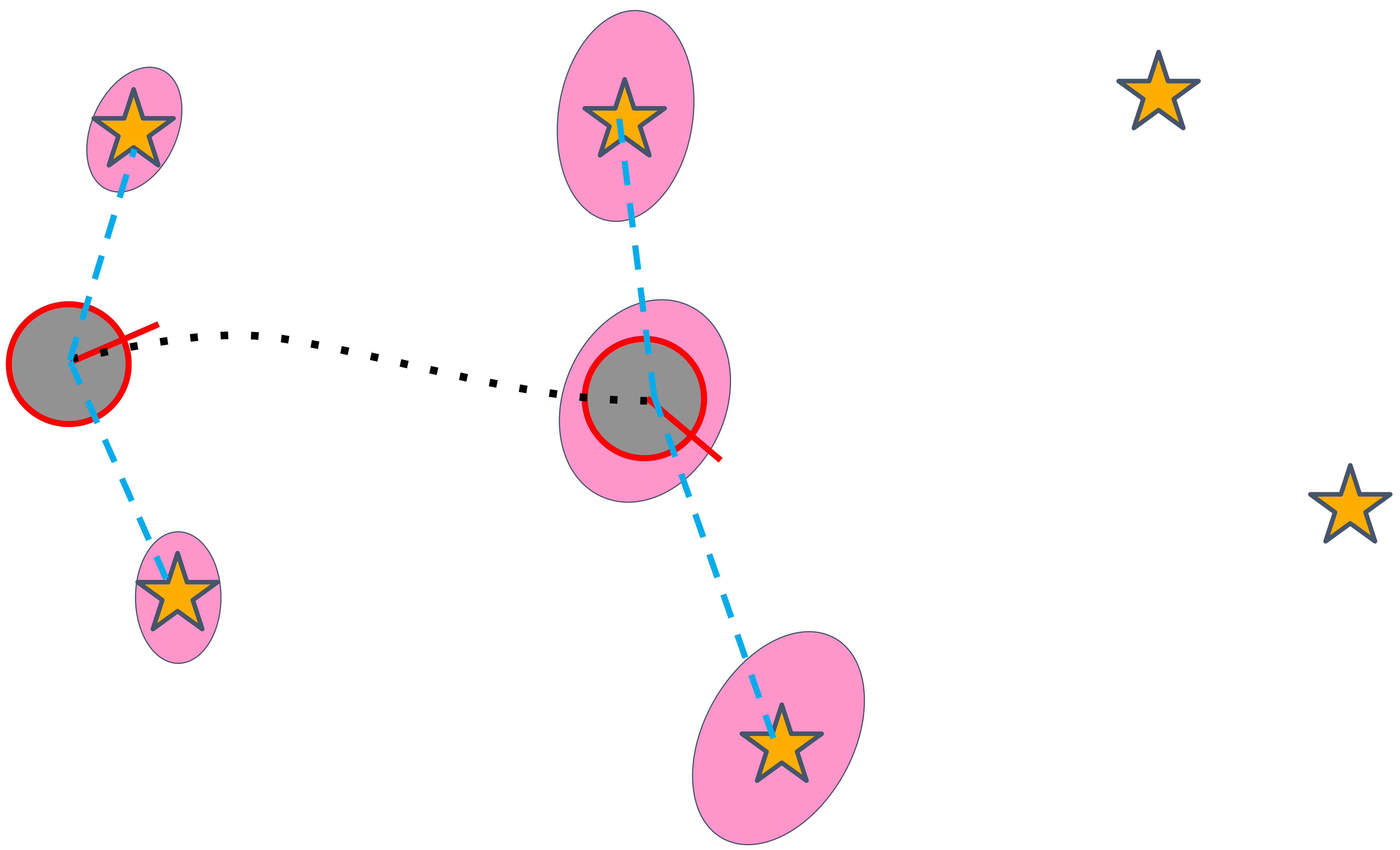

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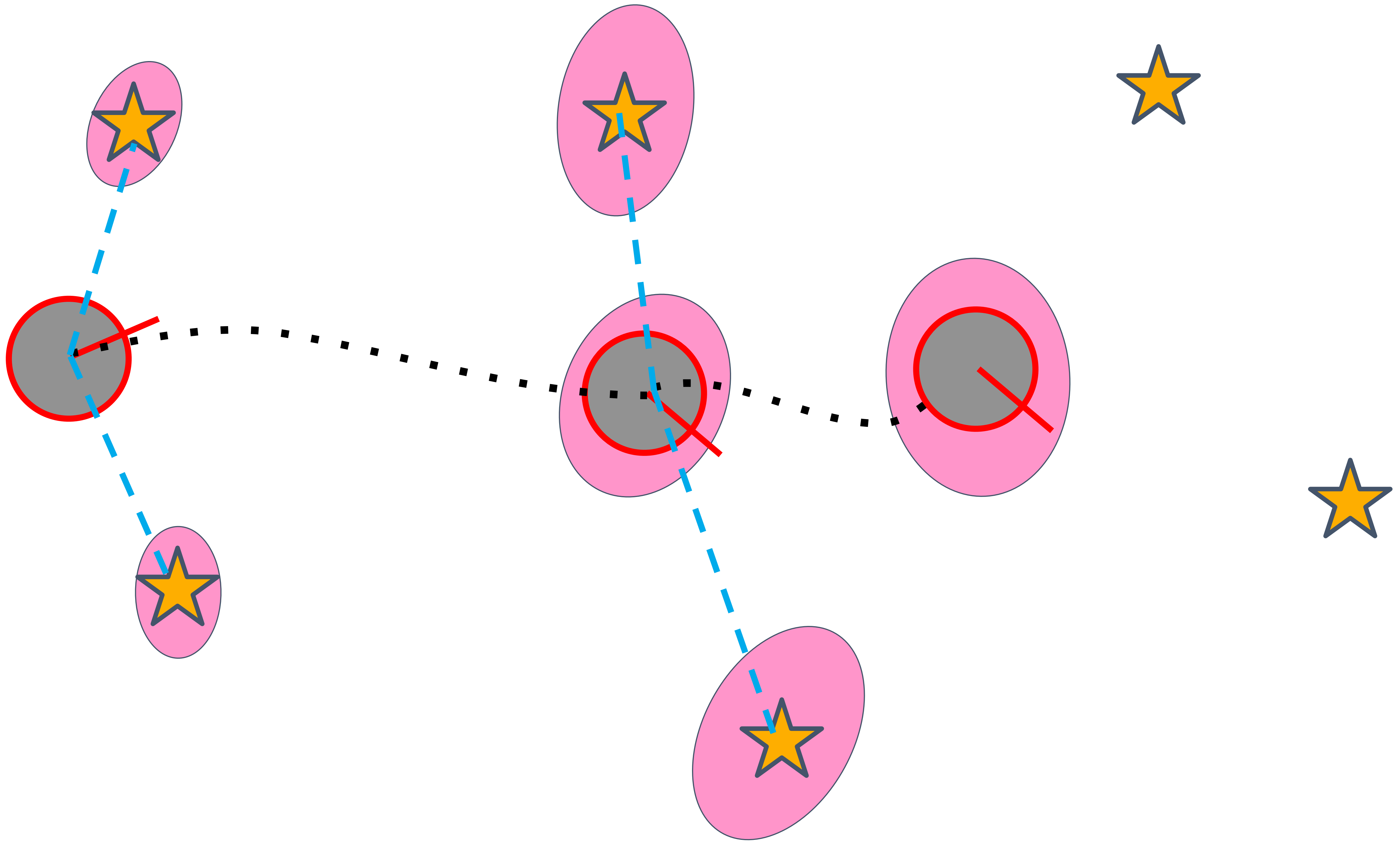

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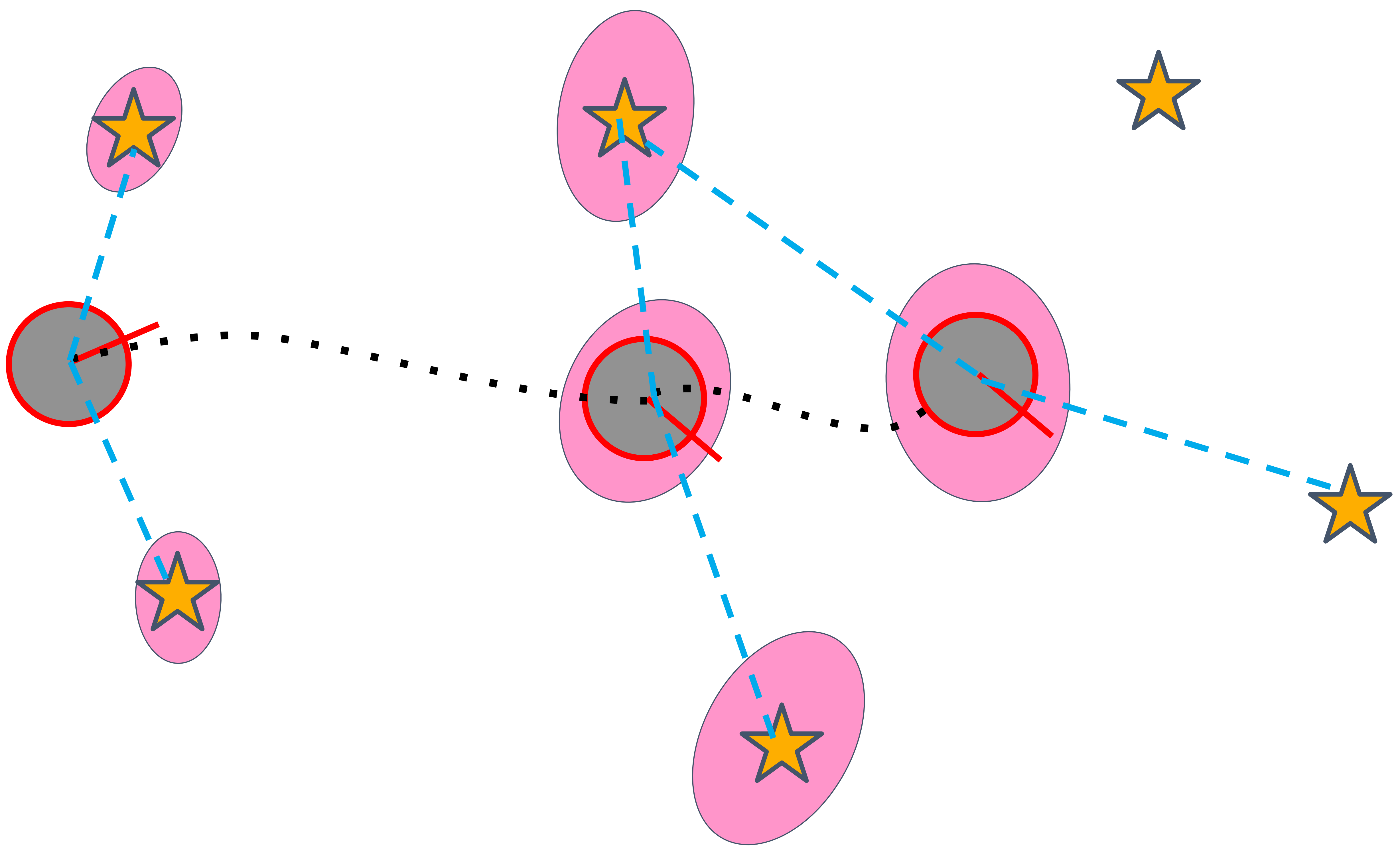
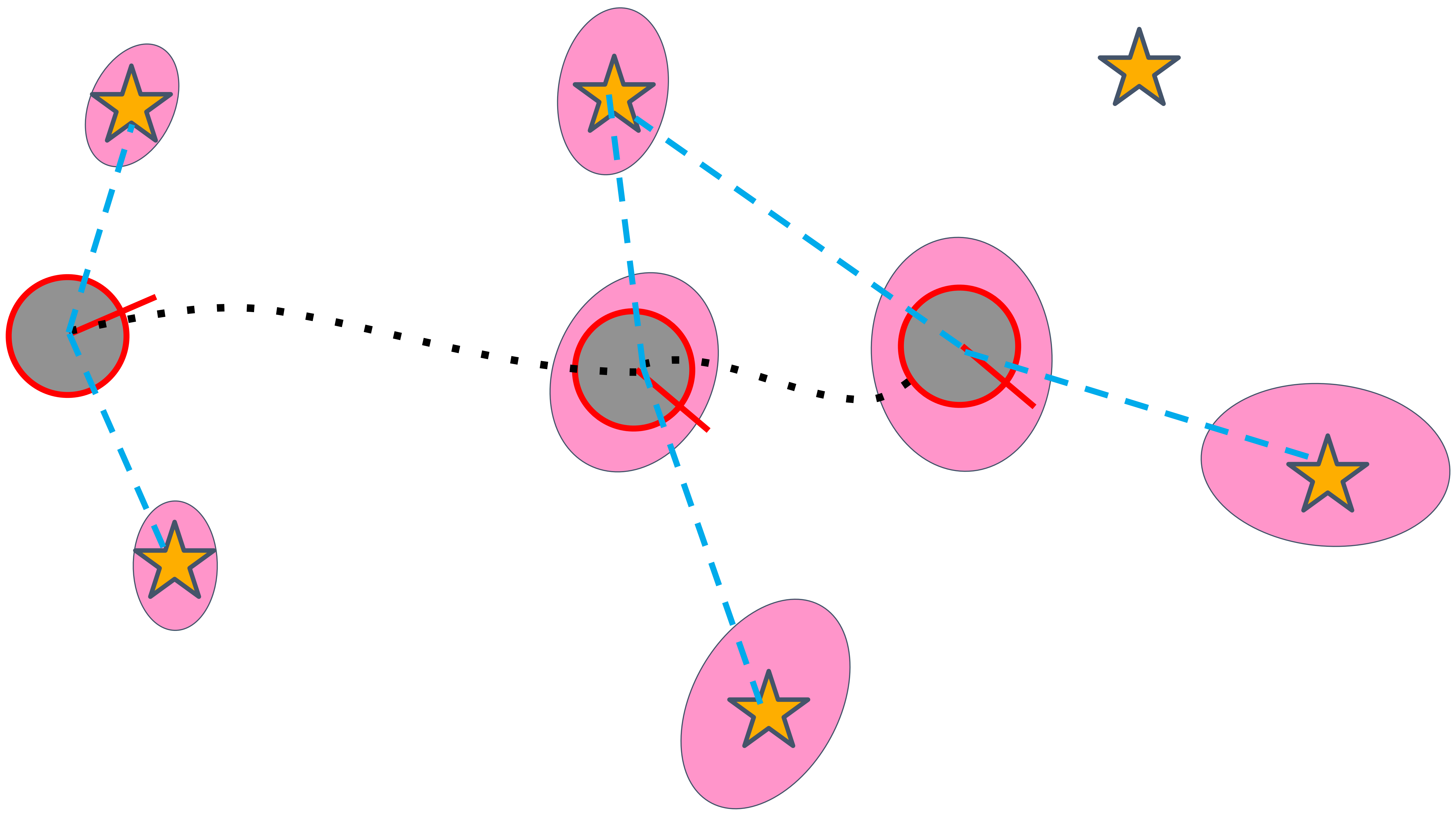


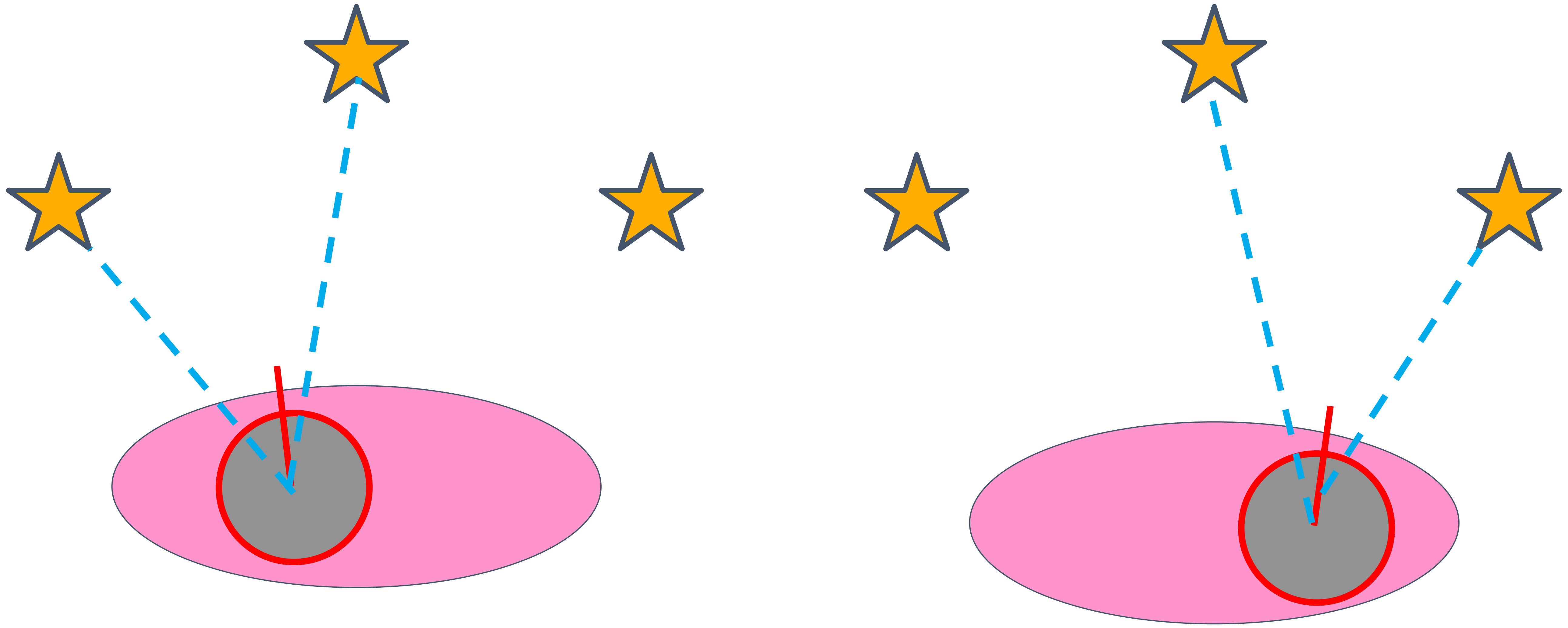
Why is SLAM hard?

- Robot pose/path and map are both unknown (not independent)
- Map and pose estimates are correlated
- Good data association is key



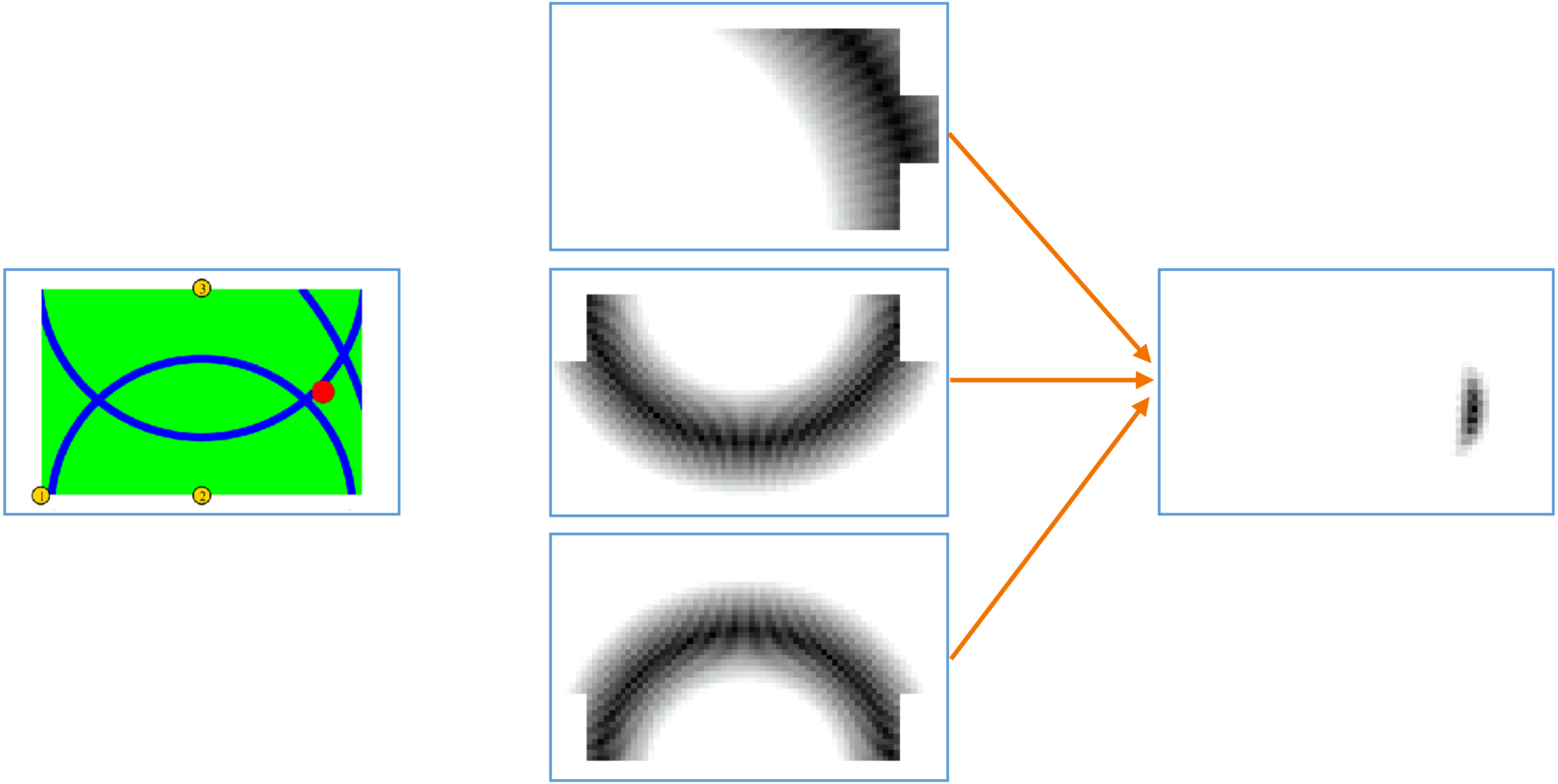
Why is SLAM hard?

- The mapping between observations and the map is unknown
- Picking the wrong data association can cause map divergence



Trilateration using range measurements

Trilateration using range measurements

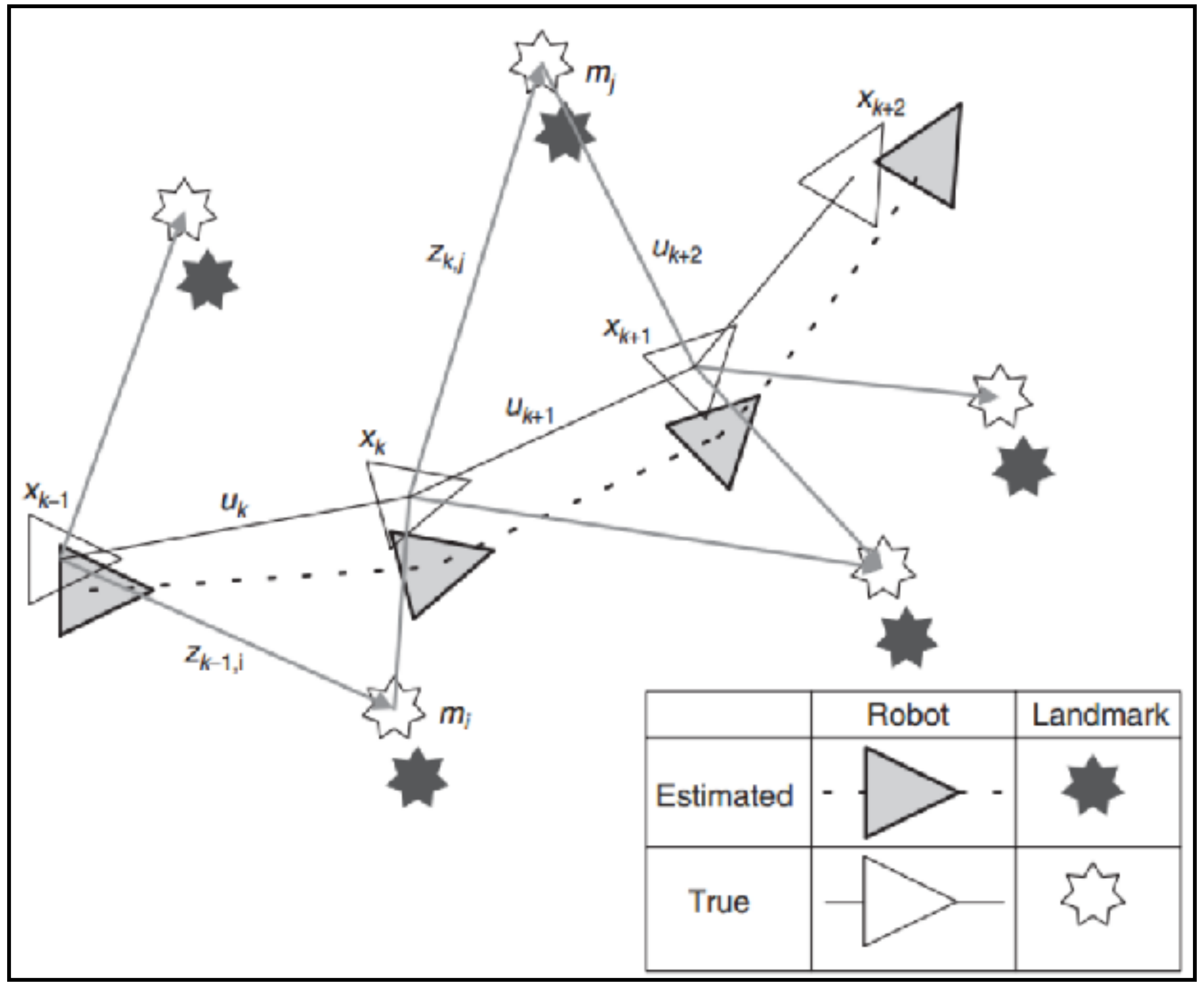


Related terms

- State estimation
- Localization
- Mapping
- **SLAM**
- Navigation
- Motion planning

- Given
  - Control inputs $U_{o:k} = \{u_1, u_2, \ldots u_k\}$
  - Relative observations $Z = \{z_1, z_2, \ldots z_n\}$
- Compute
  - Map of the environment $m = \{m_1, m_2, \ldots m_n\}$
  - Robot path (seq. of poses) $X_{0:k} = \{x_0, x_1, \ldots x_k\}$

- Error in pose
- Error in observation
- Error in mapping
- Errors accumulate

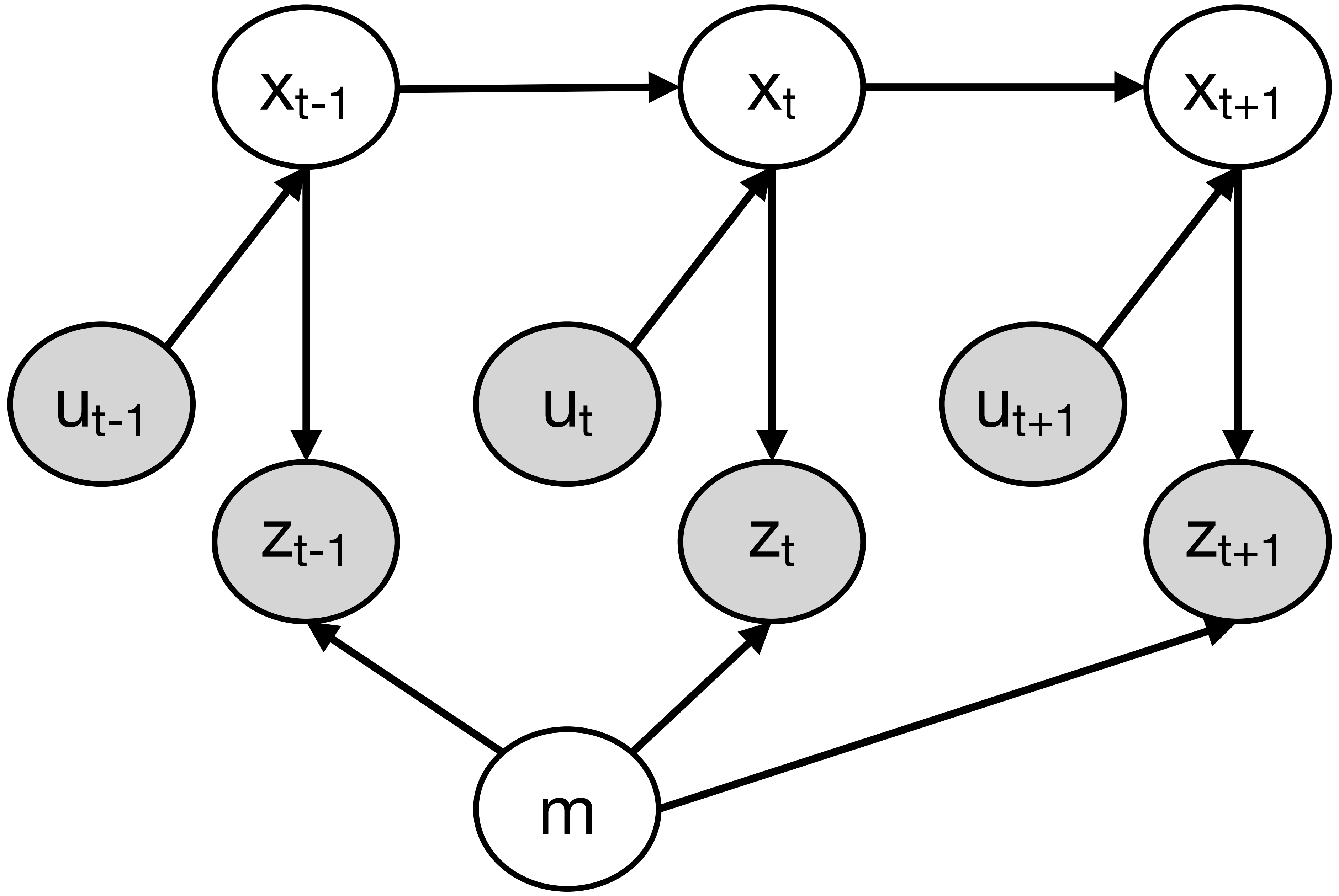


Landmarks are considered motionless

Simultaneous Localization and Mapping

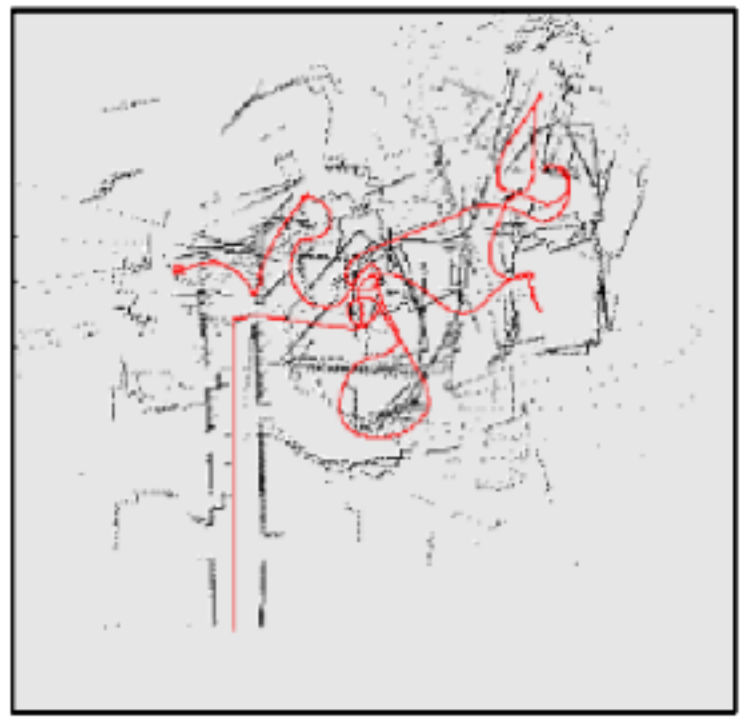
Graphical model

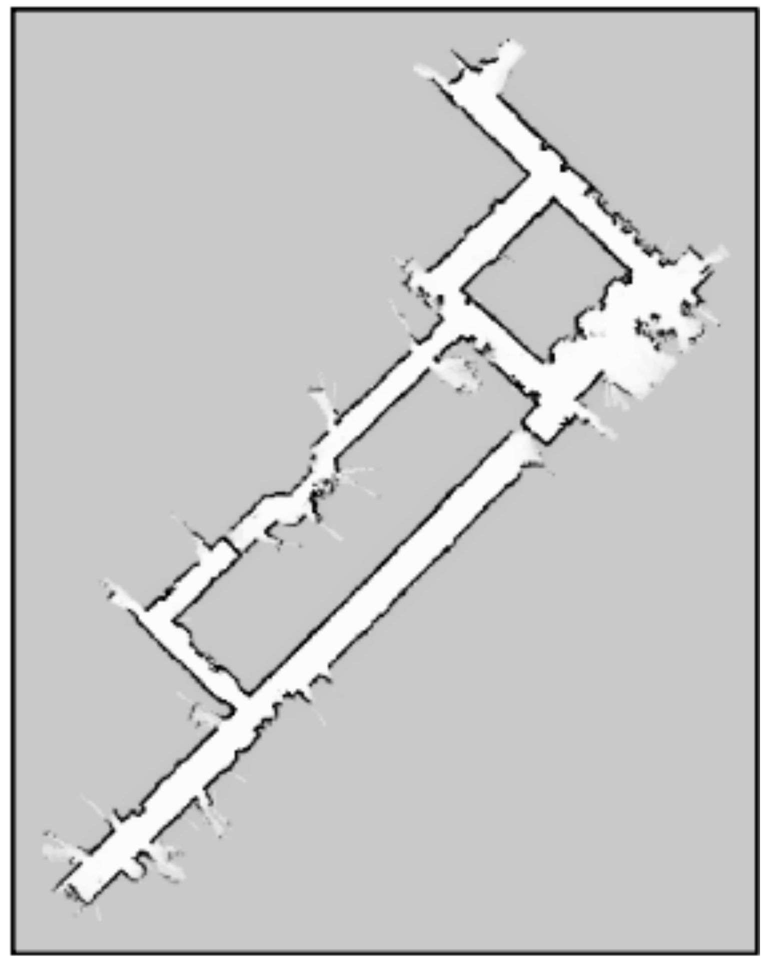
- Nodes are random variables
- Directed edges are variable dependencies
- Gray nodes: observed or directly measured variables
- White nodes: inferred latent variables



SLAM Representations

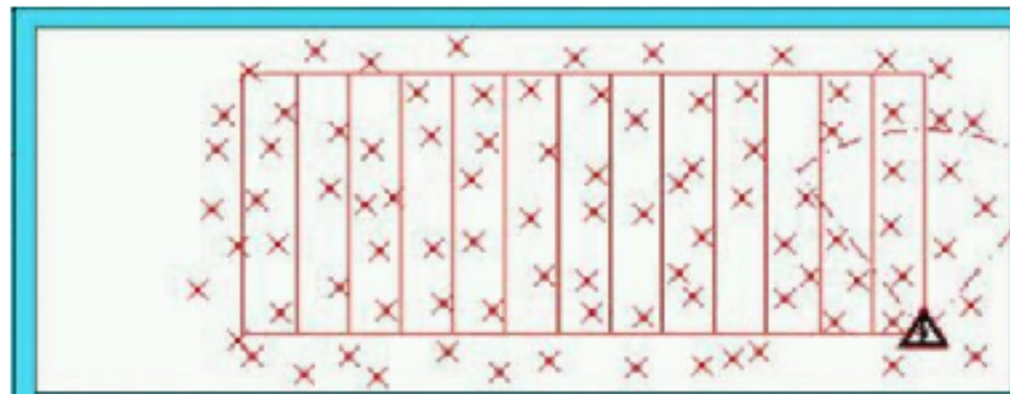
- Grid maps or scans



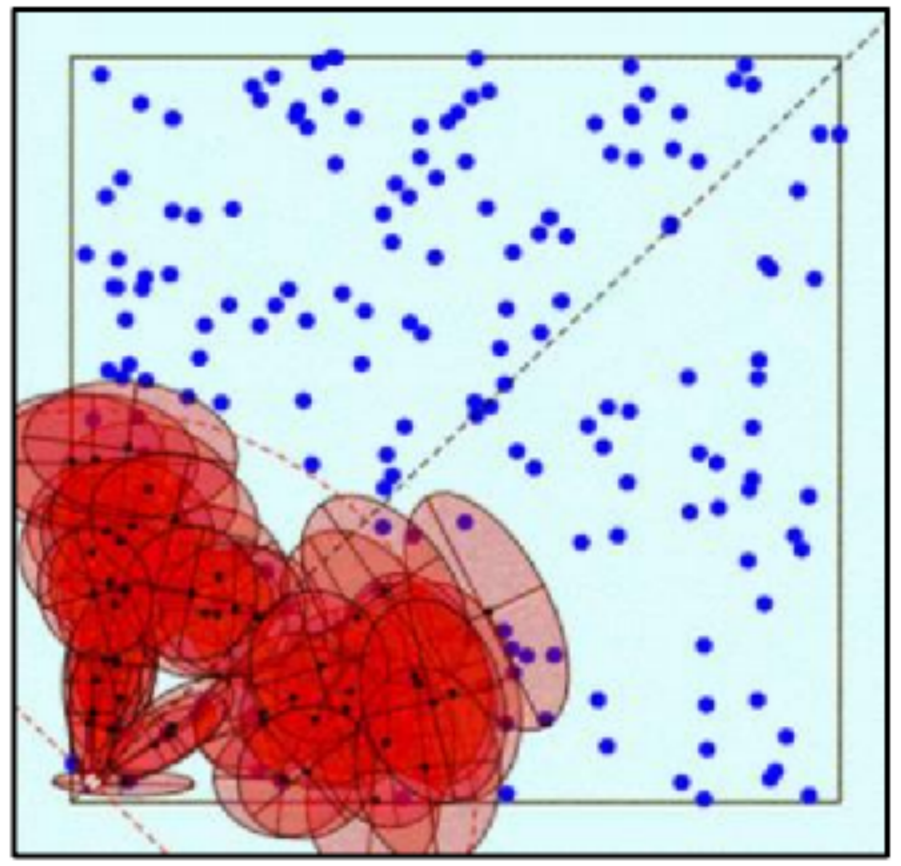


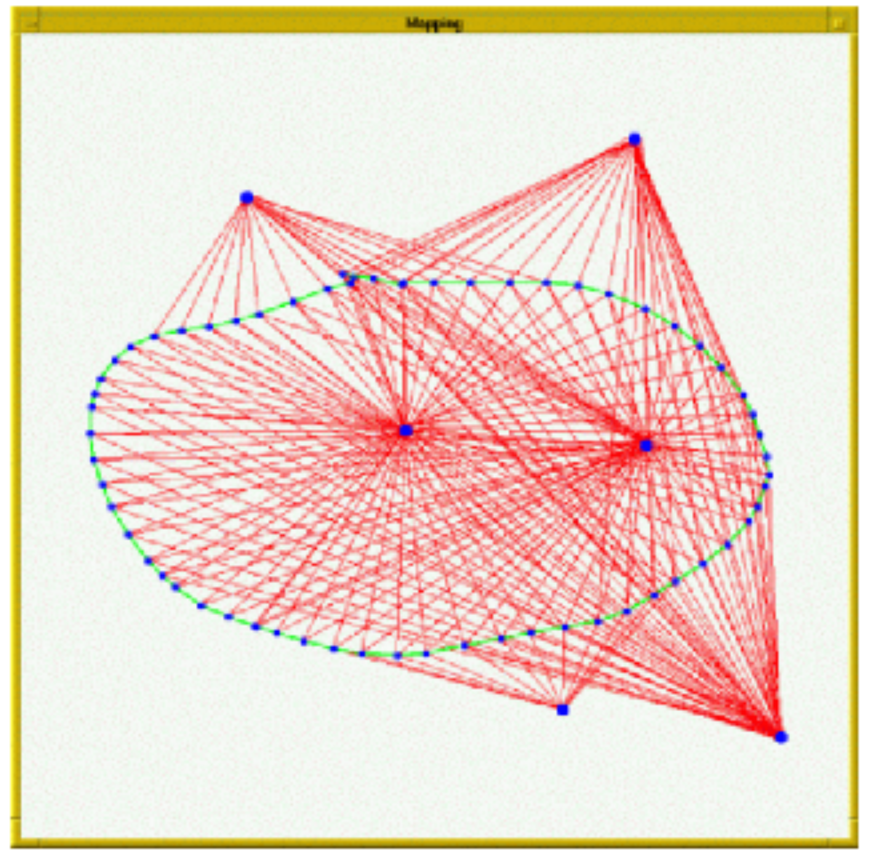
[Lu & Milios, 97; Gutmann, 98: Thrun 98; Burgard, 99; Konolige & Gutmann, 00; Thrun, 00; Arras, 99; Haehnel, 01;…]

- Landmark-based



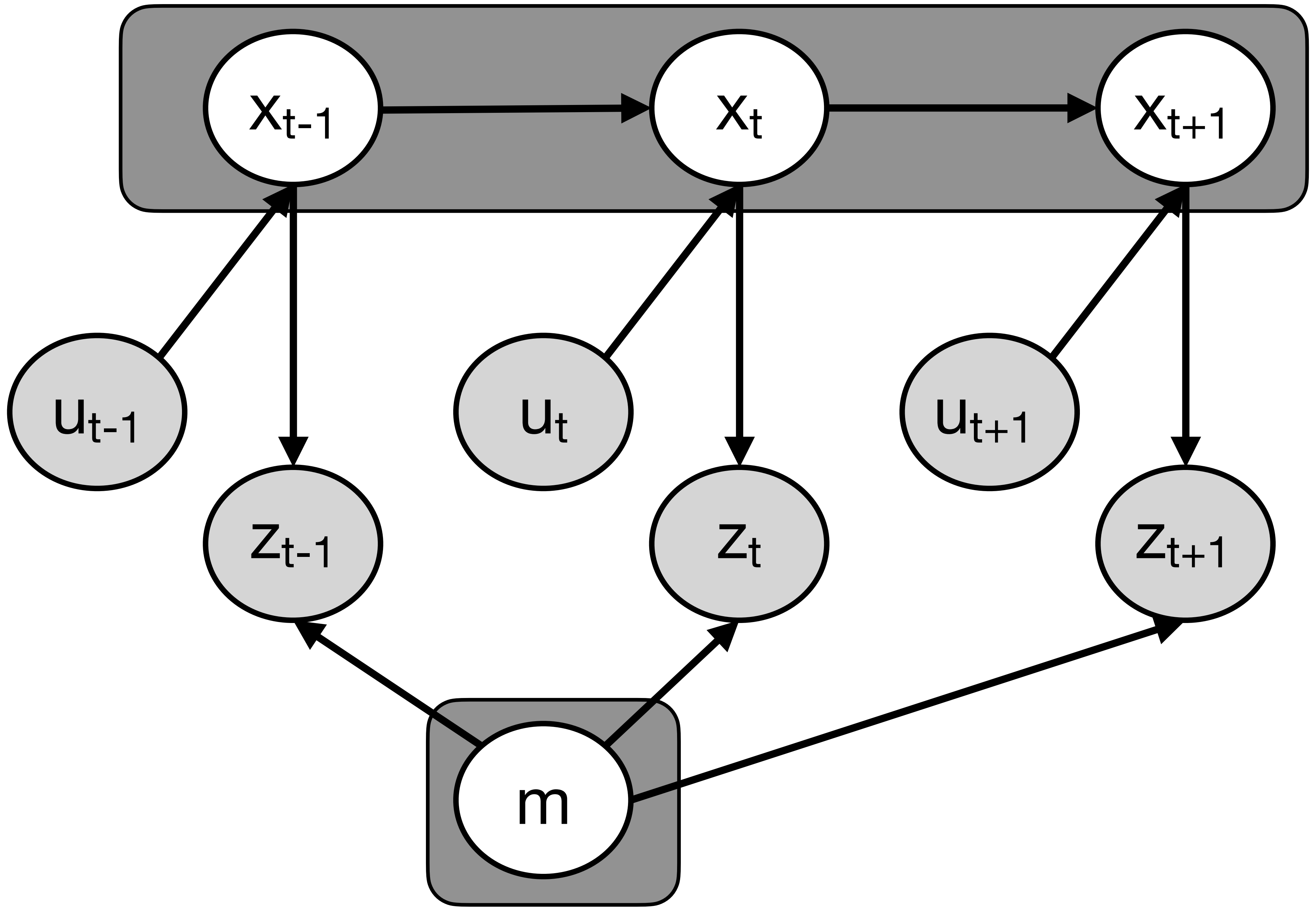
[Leonard et al., 98; Castelanos et al., 99: Dissanayake et al., 2001; Montemerlo et al., 2002;…]





Simultaneous Localization and Mapping

- Nodes are random variables
- Directed edges are variable dependencies
- Gray nodes: observed or directly measured variables
- White nodes: inferred latent variables
- Full SLAM: compute a joint posterior over the whole path of the robot and the map



$$p(x_{1:t}, m \mid z_{1:t}, u_{1:t}, x_0)$$

Simultaneous Localization and Mapping

- Nodes are random variables
- Directed edges are variable dependencies
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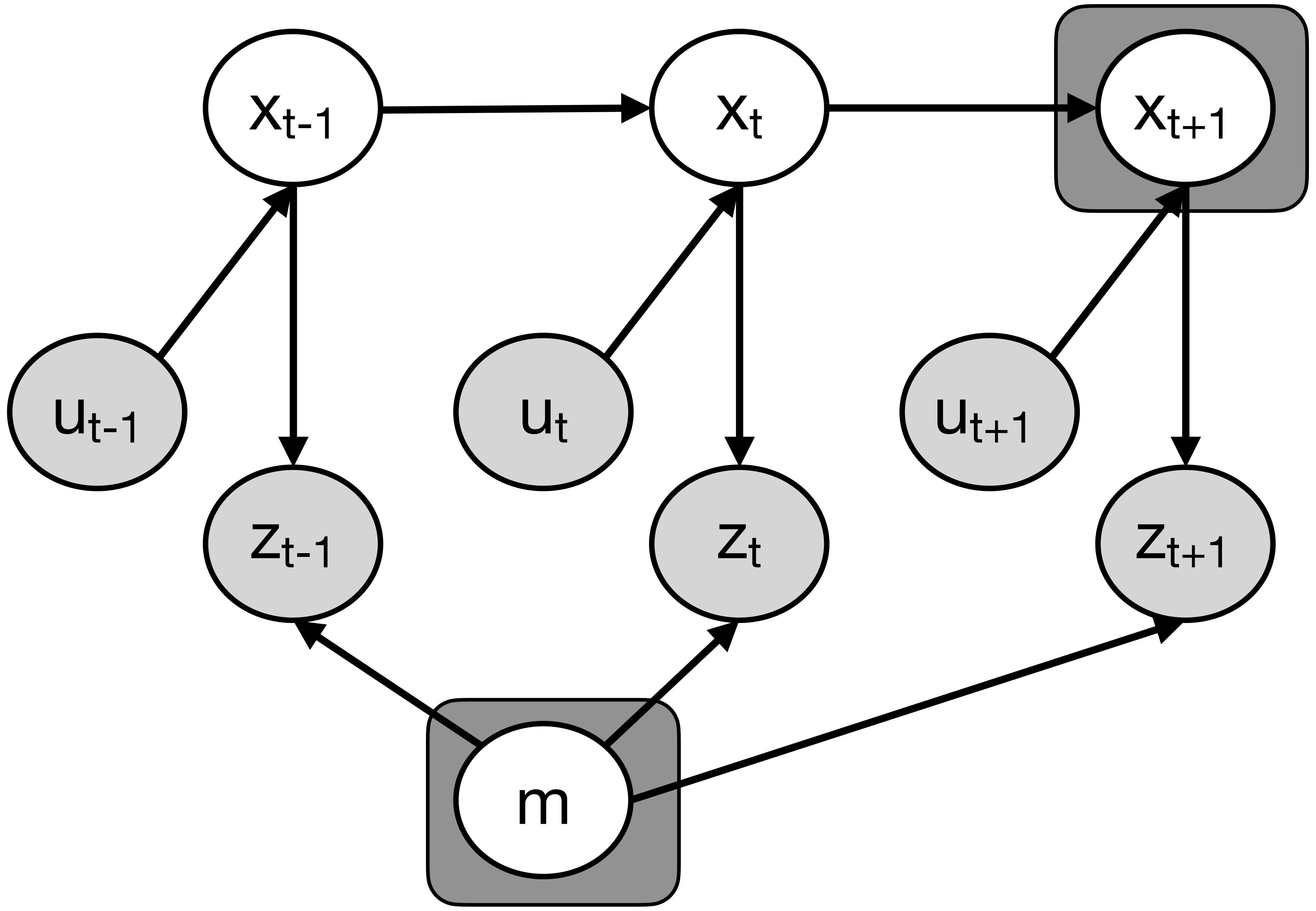
- Full SLAM: compute a joint posterior over the whole path of the robot and the map
- Online SLAM: compute a posterior over the current pose along with the map



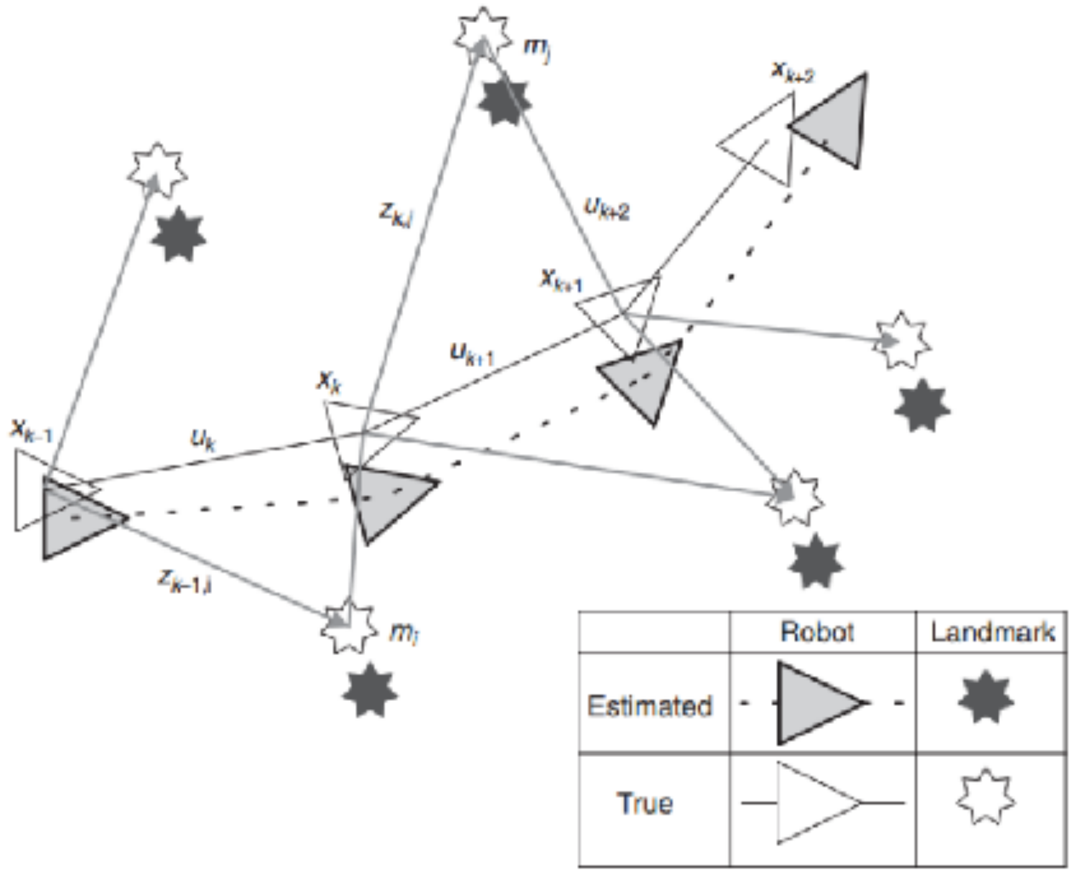
$$p(x_t, m \mid z_{1:t}, u_{1:t})$$

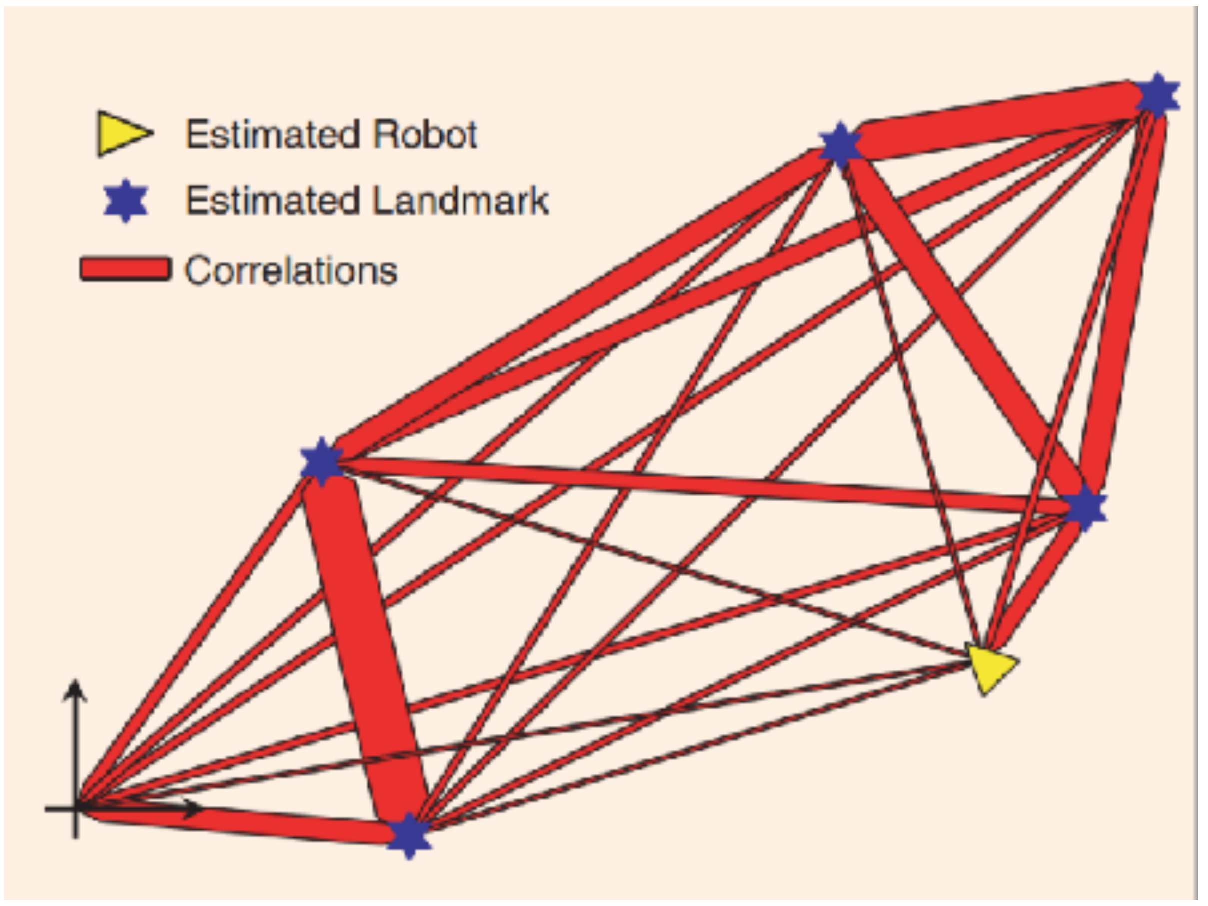
Simultaneous Localization and Mapping

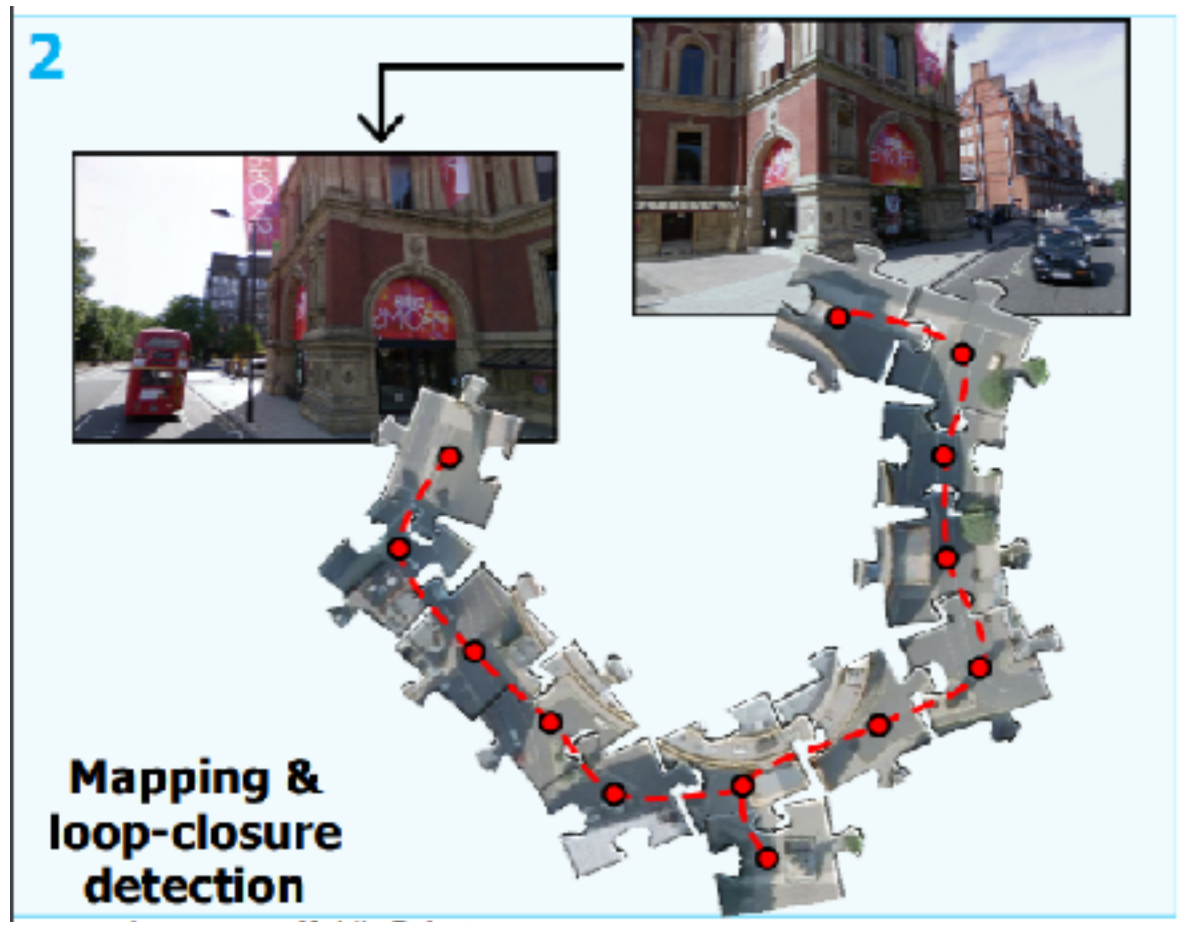
- Prediction (prediction step):
  - $p(x_t, m \mid z_{0:t}, u_{1:t}, x_0) = \Sigma_{t-1} P(x_t \mid x_{t-1}, u_{1:t}) P(x_{t-1}, m \mid Z_{0:t-1}, U_{1:t}, x_0)$
- Correction (update step):
  - $p(x_t, m \mid z_{0:t-1}, u_{0:t}, x_0) = \eta P(z_t \mid x_t, m) P(x_t, m \mid Z_{0:t}, U_{1:t}, x_0)$
- We can solve the localization problem with the assumption that we know the map
  - $P(x_t \mid Z_{0:t}, U_{0:t}, m)$
- We can solve the mapping problem with the assumption that we know the location
  - $P(m \mid X_{0:t}, Z_{0:t}, U_{0:t})$

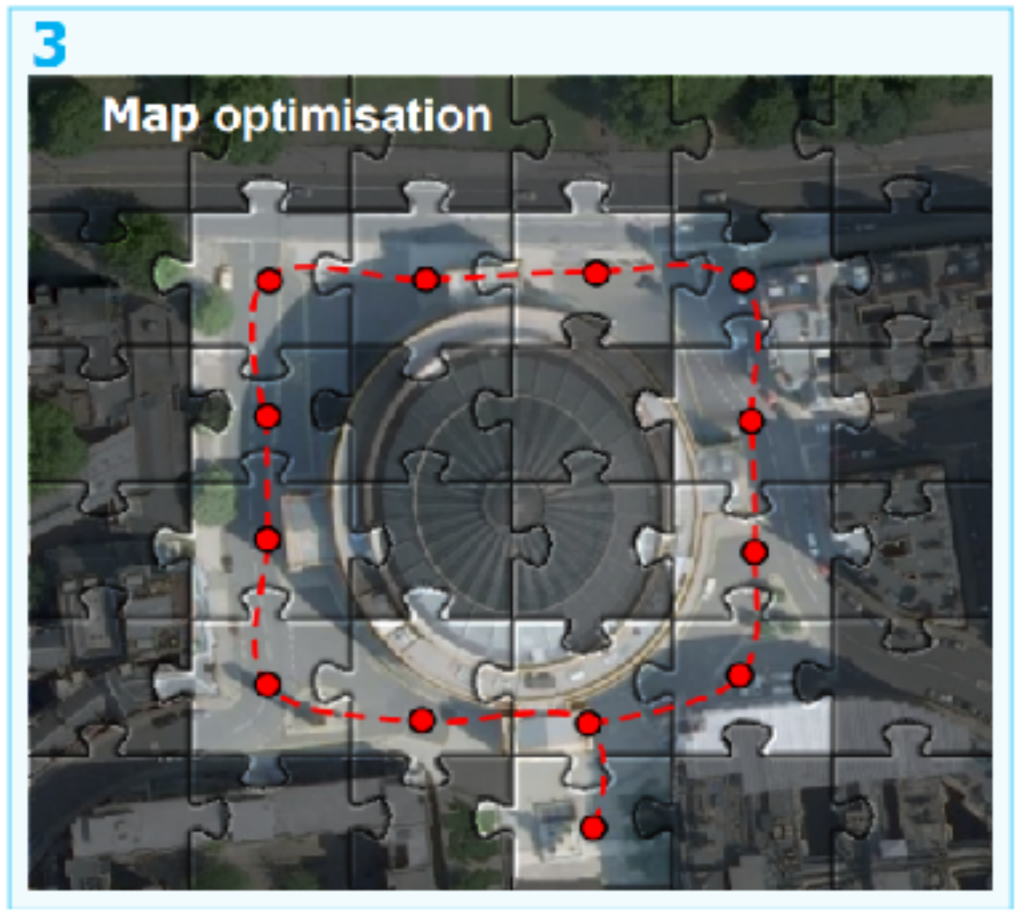
Interesting Features

- Robot observations of the **relative landmark locations can be considered nearly independent**, because the relative landmark locations are independent from the robot's coordinate frame
- Robot observations of the **absolute landmark locations are less certain**, because the absolute landmark location is strongly related to the robot's coordinate frame
- Because landmarks are correlated even unobserved landmarks can be updated, such that correlations are increased for every observation we make
- The accuracy of the relative map increases for more observations





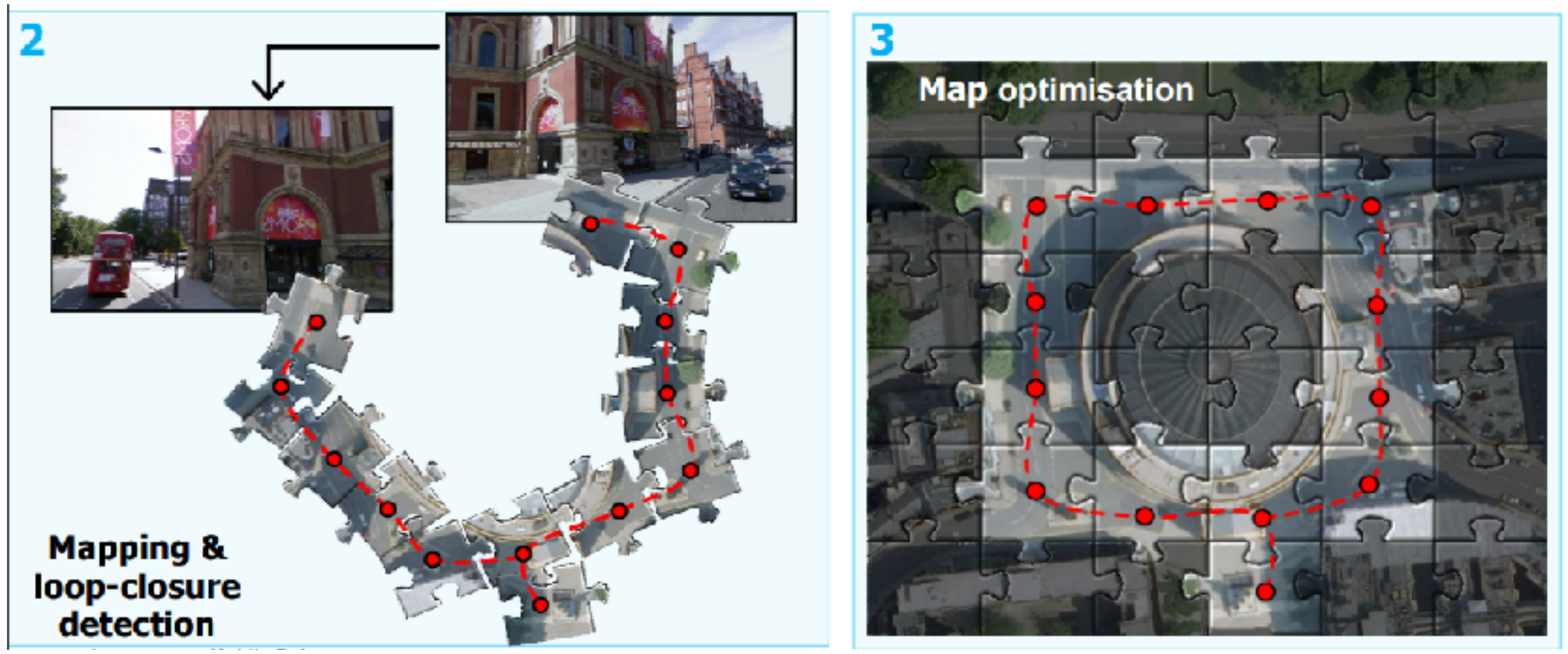




Simultaneous Localization and Mapping

Why is it hard?

- Map size
  - The larger the environment relative to the robot’s perceptual range, the more difficult it is to acquire the map
- Perceptual ambiguity
  - The more different places look alike, the more difficult it is to establish correspondence between different locations traversed at different points in time
- Cycles
  - Motion-cycles are particularly difficult to map



SLAM solutions

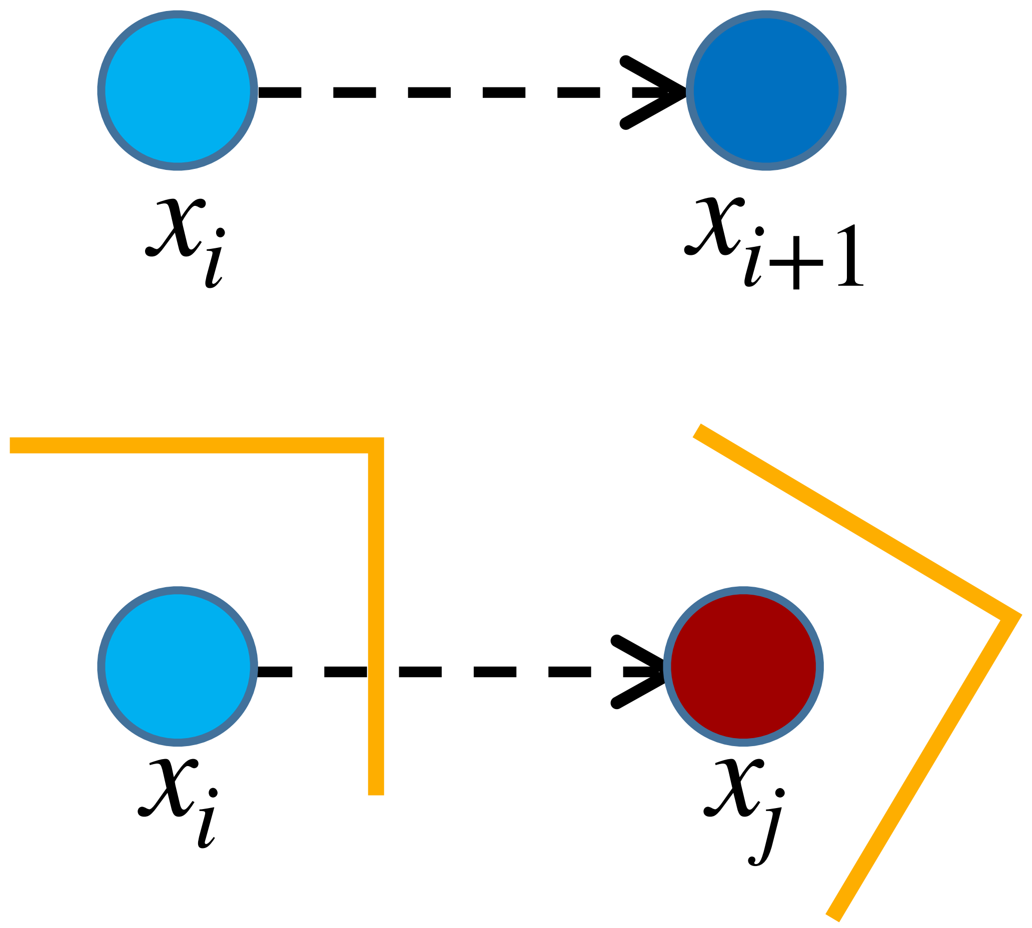
- The trick is to find an appropriate representation for the observation and the motion problem
  - Graph SLAM — Global optimization: outputs the most likely map and trajectory
  - EKF SLAM
  - Fast SLAM

  Probability distribution over landmarks and the most recent pose (online SLAM)

Graph SLAM

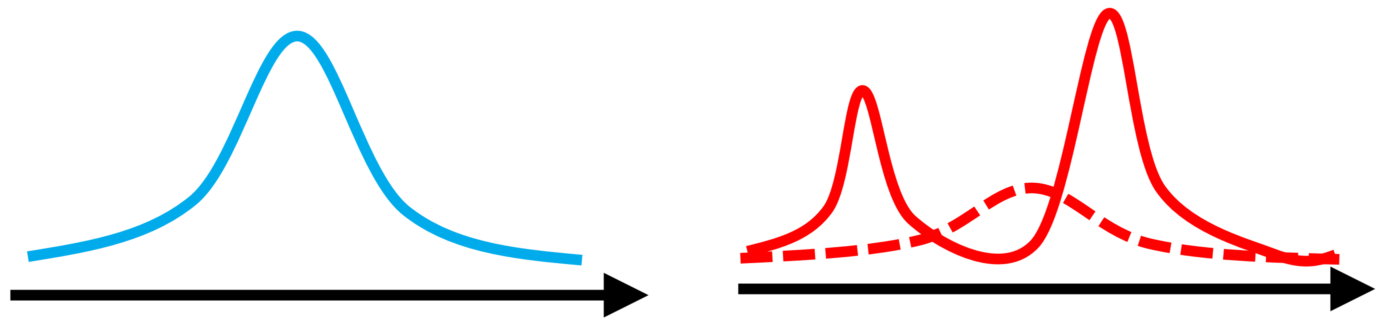
Graph SLAM

- Graph represents a set of objects where pairs of objects are connected by links encoding relations between them
- Create an edge if…
    - … the robot moves from x_i to x_{i+1}
        - edge corresponds to odometry measurement
    - … the robot observes the same part of the environment from x_i and from x_j
- Edges represent constraints
- Nodes represent the state (poses and landmarks)
    - Given a state, we can compute predicted observations
    - Find a configuration of the nodes so that the real and predicted constraints are as similar as possible
    - Minimize the Least Square Error over all constraints

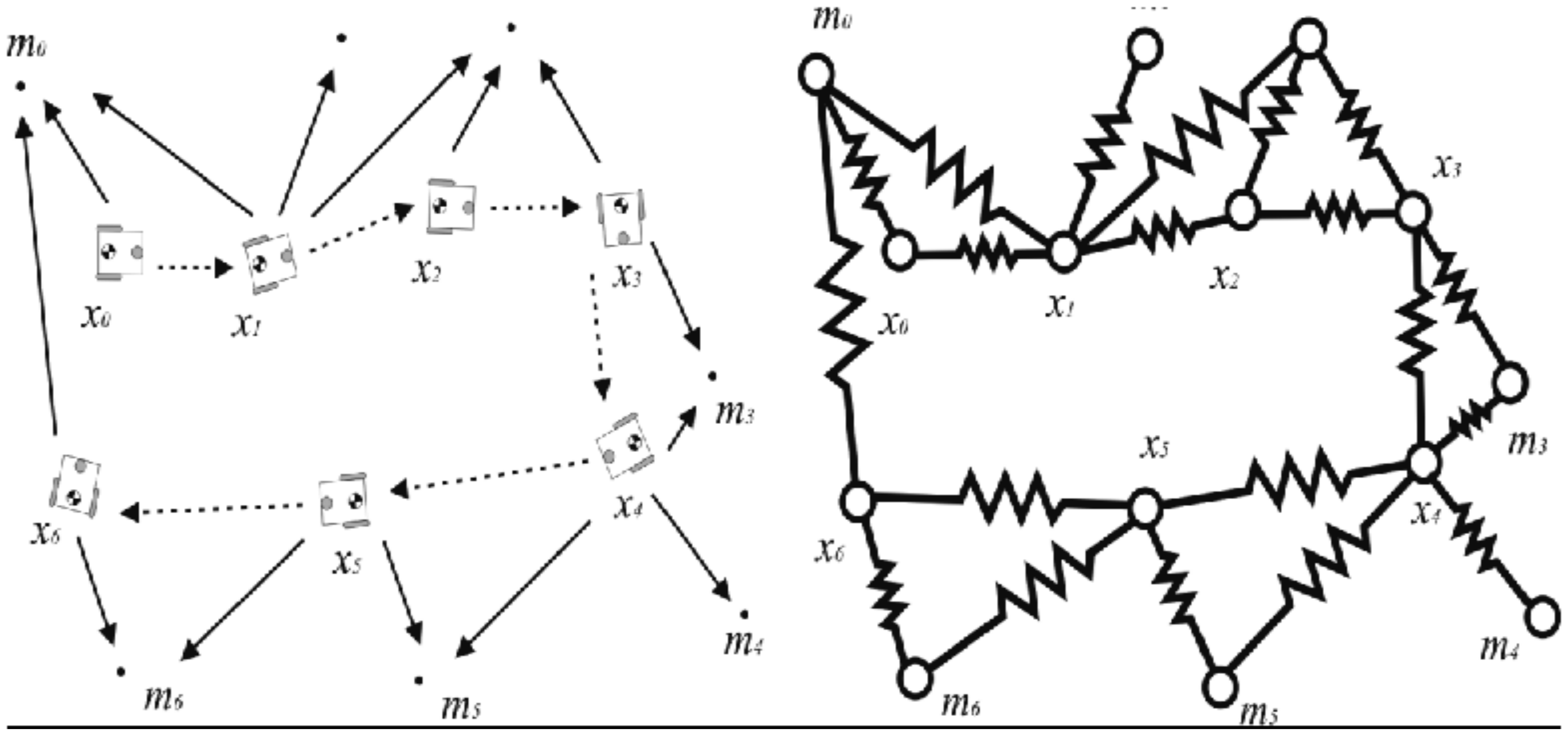


Graph SLAM

- Treat constraints (generated by motions and observations) as elastic springs
- Minimize the energy in all the springs
- Any modern SLAM implementation has some version of this
  - Pro: globally optimal
  - Con: BIG optimization problem, only one output



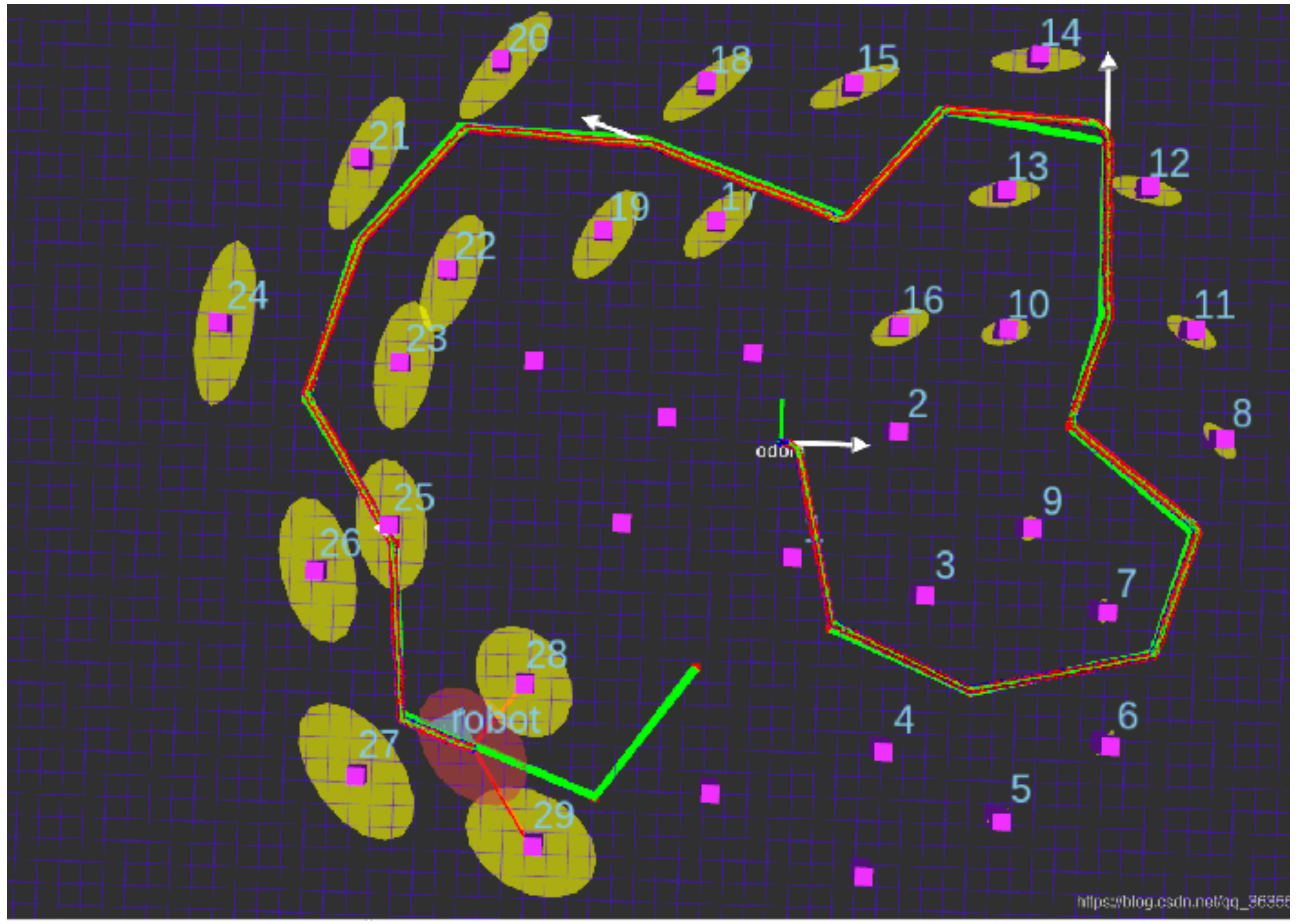
- Tricks
  - Combine poses over many time steps into single nodes to make the graph smaller
  - If you see the same landmark from several poses, you can get rid of the pose and add a stronger constraint between those landmarks



EKF SLAM

EKF SLAM

- Goal: Estimate $p(x_k, m \,|\, u_{1:k}, z_{1:N})$
- Assume all noise is Gaussian
- Track a Gaussian belief of the current state and landmarks
- Apply the Kalman Filter…



Kalman Filter

Kalman Filter ($\mu(t-1)$, $\Sigma(t-1)$, $u(t)$, $z(t)$)

1. $\mu_p(t) = A\mu(t-1) + Bu(t)$
2. $\Sigma_p(t) = A\Sigma(t-1)A^T + \Sigma_u$
3. $K_{KF} = \Sigma_p(t)C^T(C\Sigma_p(t)C^T + \Sigma_z)^{-1}$
4. $\mu(t) = \mu_p(t) + K_{KF}(z(t) - C\mu_p(t))$
5. $\Sigma(t) = (I - K_{KF}C)\Sigma_p(t)$
6. Return $\mu(t)$ and $\Sigma(t)$

prediction (steps 1–2)

update (steps 3–5)

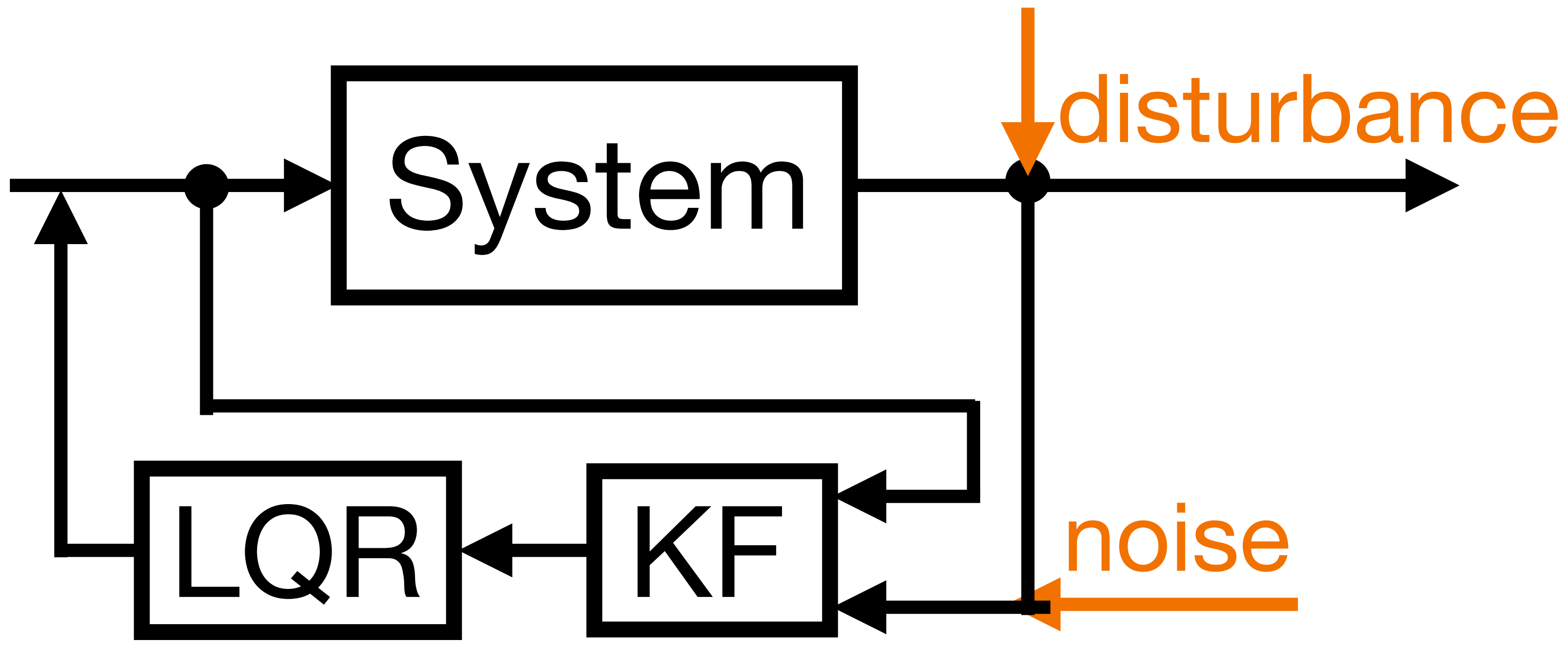
State estimate: $\mu(t)$

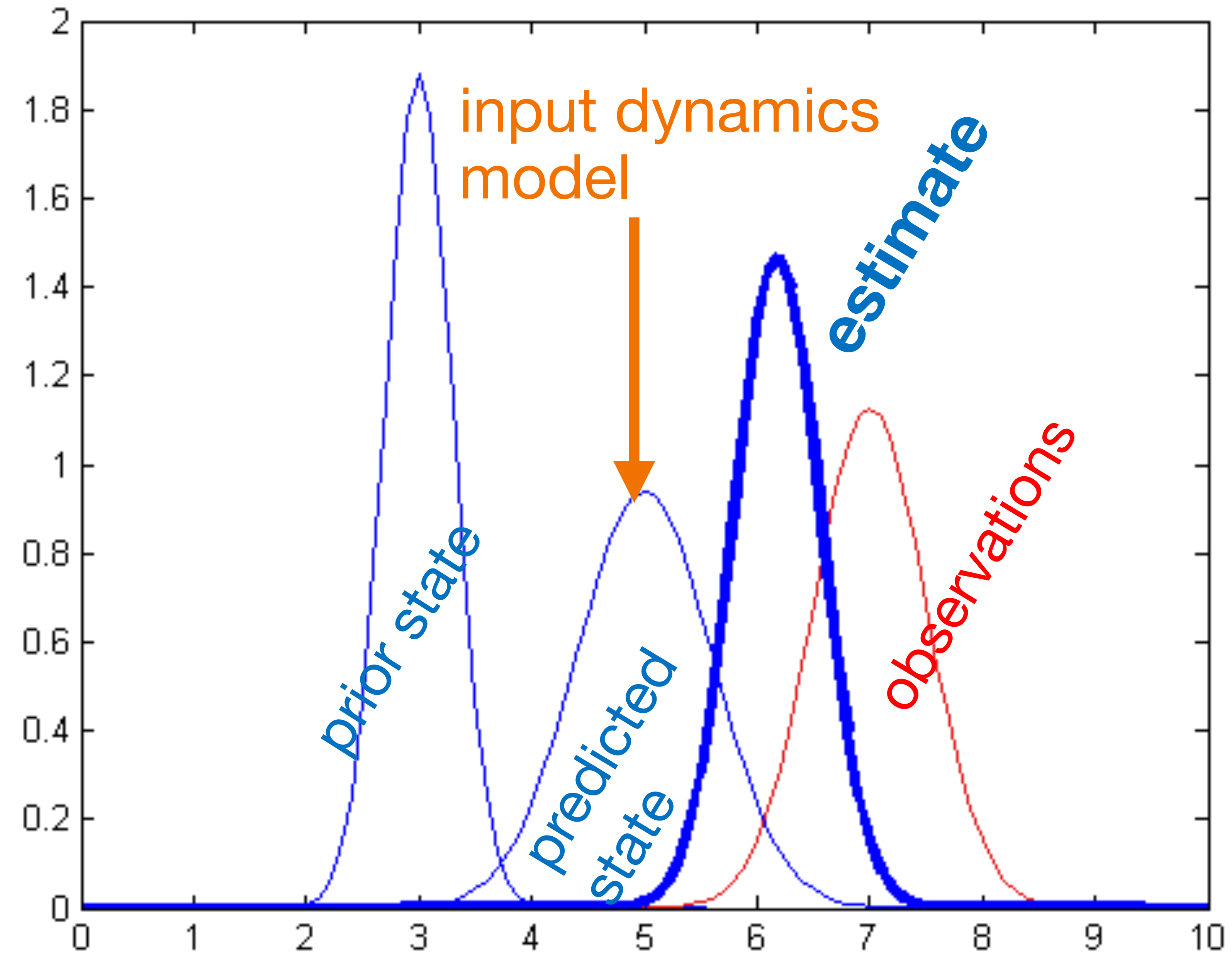
State uncertainty: $\Sigma(t)$

Process noise: Σ_u

Kalman filter gain: K_{KF}

Measurement noise: Σ_z





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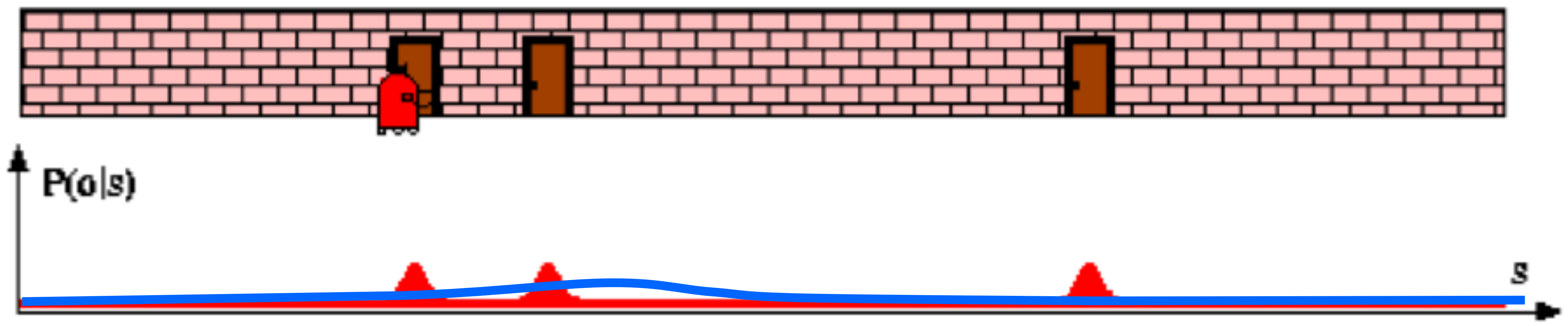
- Goal: Estimate $p(x_k, m \mid u_{1:k}, z_{1:N})$
- Assume all noise is Gaussian
- Track a Gaussian belief of the current state and landmarks
- Linearize around every state and run the Kalman Filter

$$x = \begin{bmatrix} \bar{\wp} \\ \bar{\mathscr{M}} \end{bmatrix} = \begin{bmatrix} \wp \\ \mathscr{L}_1 \\ \vdots \\ \mathscr{L}_n \end{bmatrix} \qquad P = \begin{bmatrix} P_{\wp\wp} & P_{\wp\mathscr{M}} \\ P_{\mathscr{M}\wp} & P_{\mathscr{M}\mathscr{M}} \end{bmatrix} = \begin{bmatrix} P_{\wp\wp} & P_{\wp\mathscr{L}_1} & \cdots & P_{\wp\mathscr{L}_n} \\ P_{\mathscr{L}_1\wp} & P_{\mathscr{L}_1\mathscr{L}_1} & \cdots & P_{\mathscr{L}_1\mathscr{L}_n} \\ \vdots & \vdots & \ddots & \vdots \\ P_{\mathscr{L}_1\wp} & P_{\mathscr{L}_n\mathscr{L}_1} & \cdots & P_{\mathscr{L}_n\mathscr{L}_n} \end{bmatrix}$$

- Landmark matrix grows, making the inversion step costly!
- In Full SLAM the trajectory matrix grows even faster

EKF SLAM

- Goal: Estimate $p(x_k, m \mid u_{1:k}, z_{1:N})$
- Assume all noise is Gaussian
- Track a Gaussian belief of the current state and landmarks
- Linearize around every state and run the Kalman Filter
- Pros
  - Super easy, well understood, runs online
  - Works well for low-uncertainty problems



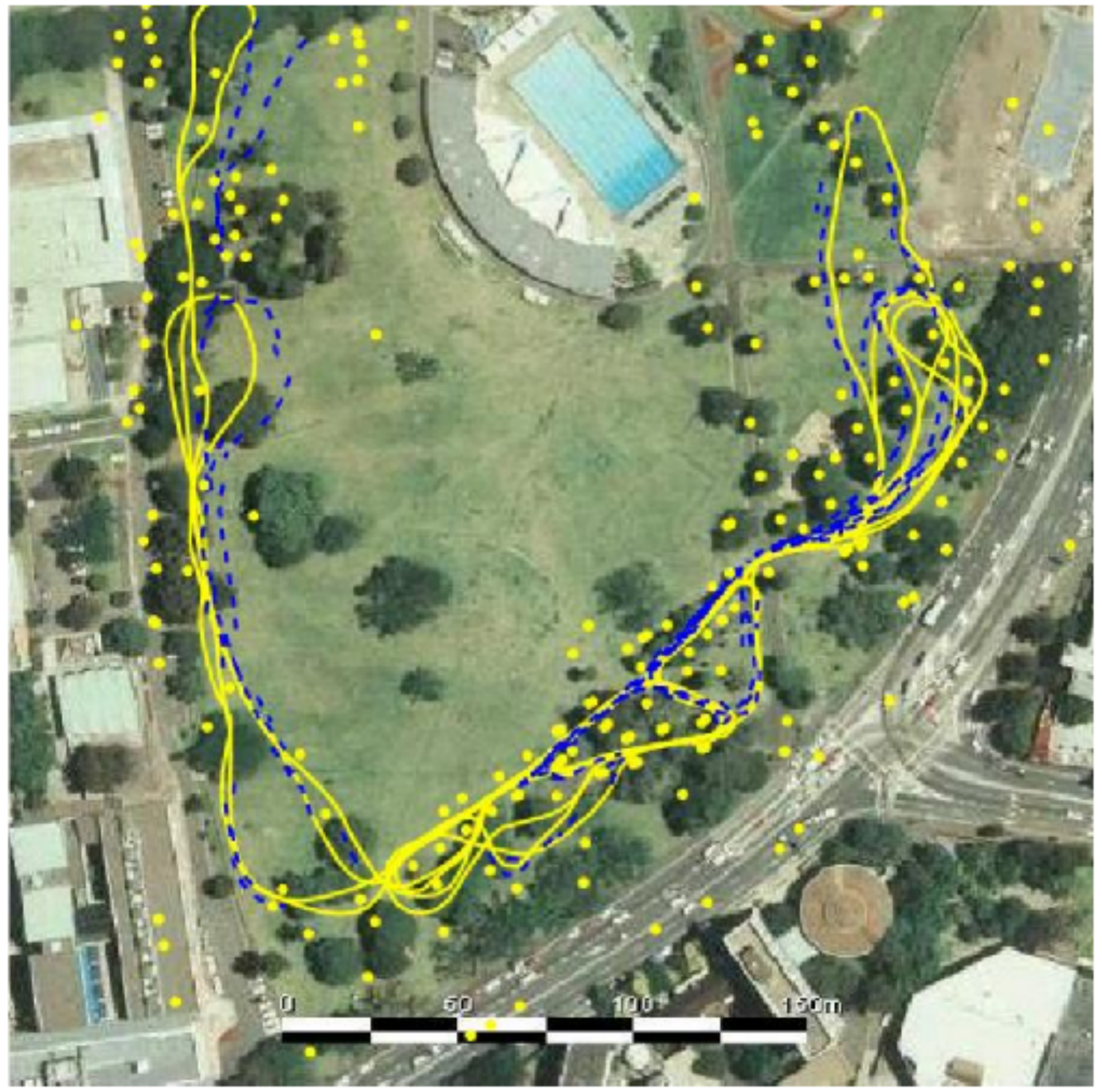
- Cons
  - Works poorly for high-uncertainty problems
  - States must be well-approximated by Gaussians

Fast SLAM

Fast SLAM

- Half sample-based solution
  - Particle filter
    - Every particle has its own version of the map with a given trajectory
- Half analytical solution
  - Landmark-based
    - Each pose and map of independent features is updated analytically through EKF
  - Grid-based map
    - Occupancy of each grid cell is estimated by Bayes Filter

Victoria Park dataset
University of Sydney

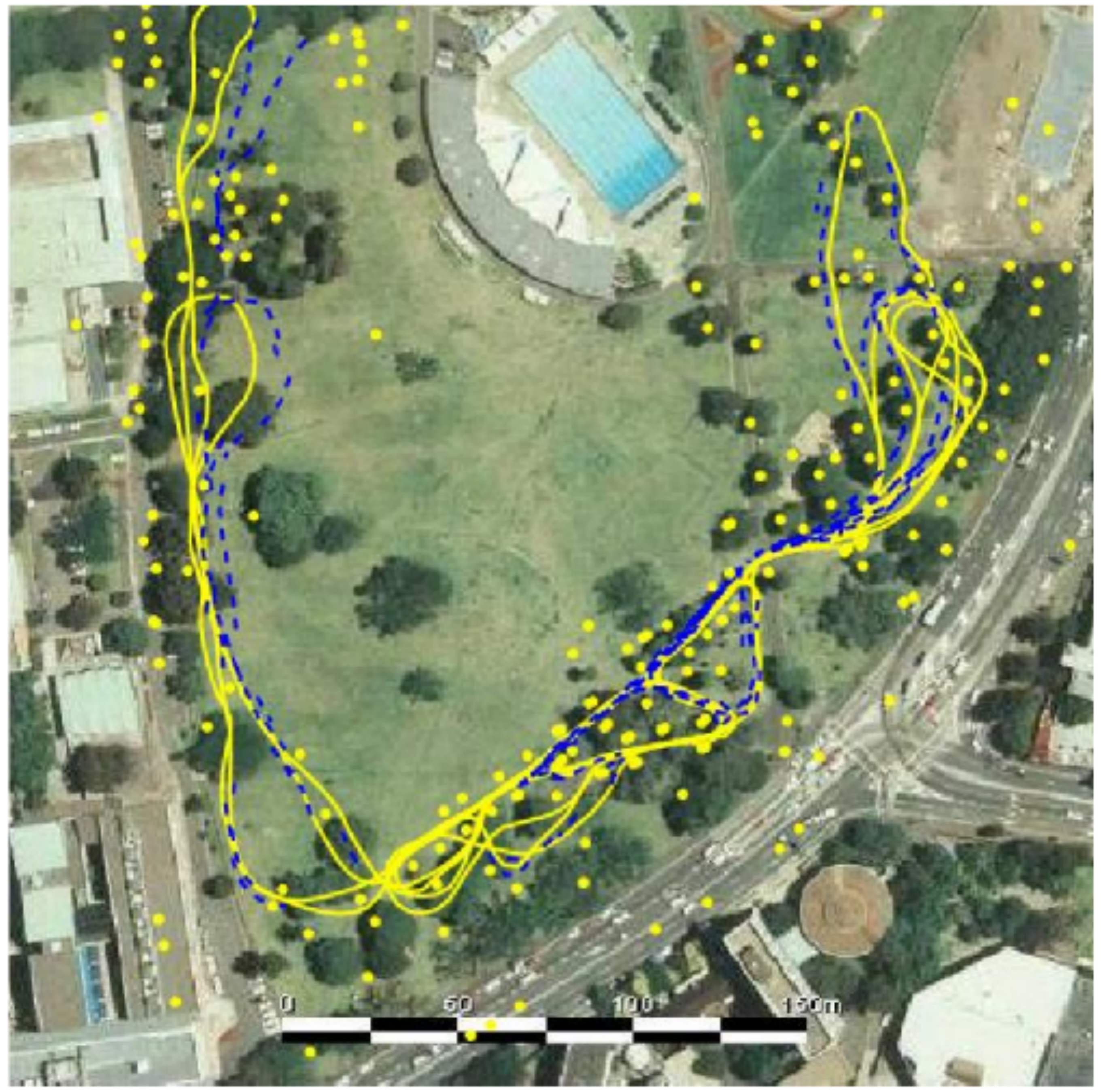


GPS
FastSLAM
4km traverse
100 particles
<5m RMS position error

Fast SLAM

- Key idea: factorize the posterior
  - $p(x_{1:k}, m \mid z_{1:k}) = p(m \mid x_{1:k}, z_{1:k}) p(x_{1:k} \mid z_{1:k})$
- $p(x_{1:k} \mid z_{1:k})$: pose estimation is approximated by the Particle Filter (can represent multiple hypotheses)
- $p(m \mid x_{1:k}, z_{1:k})$: classic mapping problem, approx using EKF (efficient at representing belief in high dimensions)
- Outcome is a Marginalized Particle Filter (MPF)
  - Each particle is a pose trajectory with an attached map corresponding to mean and covariance of each landmark

Victoria Park dataset
University of Sydney

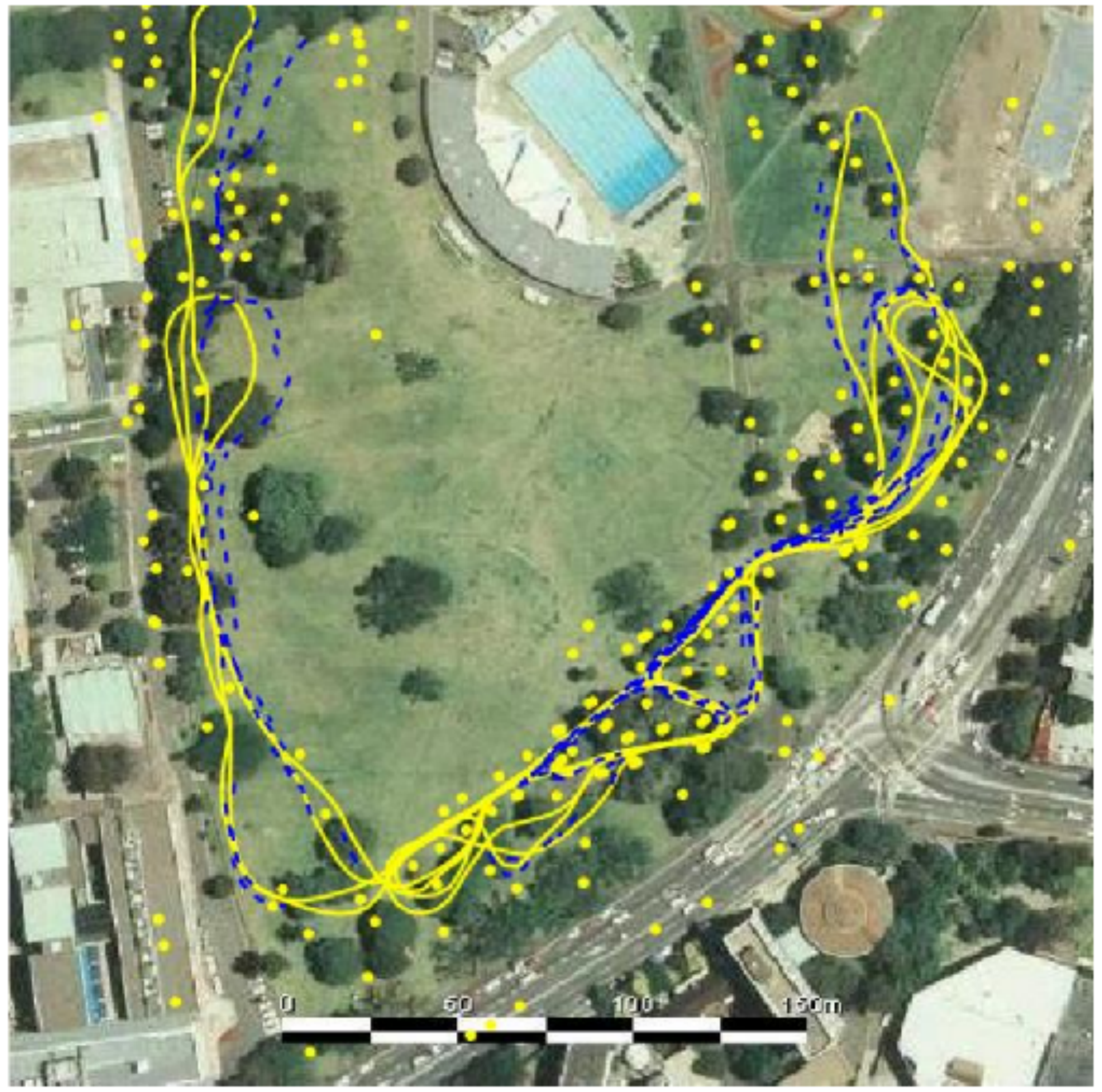


GPS
FastSLAM
4km traverse
100 particles
<5m RMS position error

Fast SLAM

- Distribution is estimated by a fixed number of particles
  - Each particle, k, contains an estimate of robot path and the mean and covariance of each of the n features
  - $P^{[k]}(x_t^{[k]}; \mu^{[k]}, \Sigma_1^{[k]}; \ldots \mu^{[k]}, \Sigma_n^{[k]})$
- **Step 1:** Update particle trajectory (motion model)
- **Step 2:** Update particle landmarks with EKF (sensor model)
  - Linearize the observation model at $(x_t^{[k]}, m)$
  - Only update associated landmarks
- **Step 3**: Update weights based on $p(z_t \mid x_t^{[k]}, m^{[k]})$
- **Step 4:** Resample distribution

Victoria Park dataset
University of Sydney



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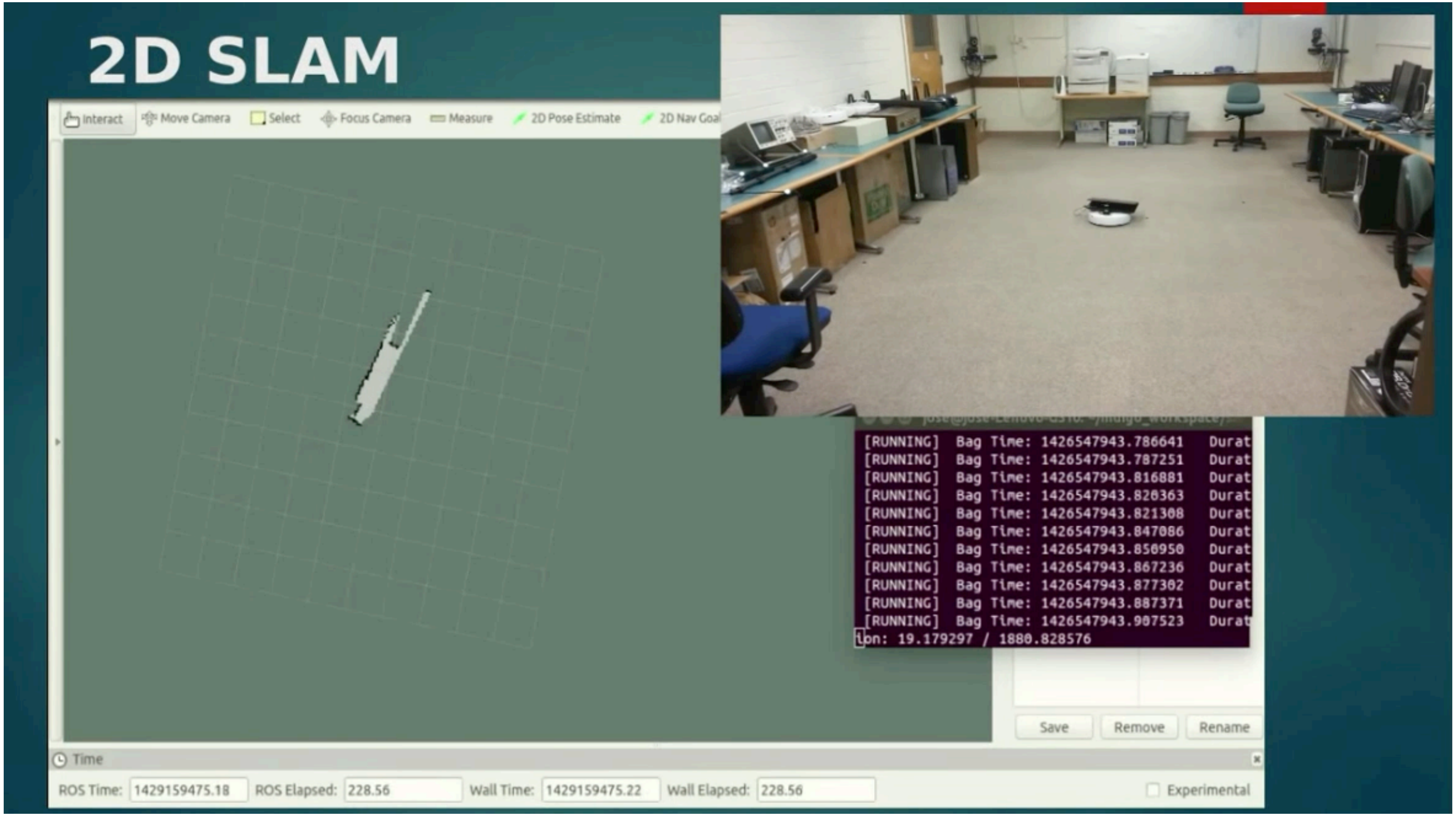
SLAM State of the Art

State of the Art in SLAM



https://www.youtube.com/watch?v=6pRAhfBMW8w

State of the Art in SLAM

https://www.youtube.com/watch?v=ufvPS5wJAx0

State of the Art in SLAM

- Robotics
  - 3D cameras with depth maps and high frame rates and resolution
  - Dense 3D models of the world
  - Uses ROS and deep learning to recognize features
  - Come built-in in a range of robots
  - Inherent to e.g. the RealSense tracking cameras
- 3D scanning/ reconstruction
- Virtual and augmented reality

State of the Art in SLAM