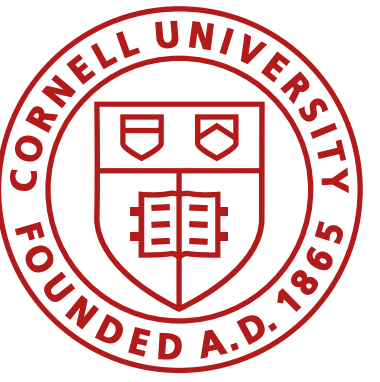


Optimal Control

Fast Robots, ECE4160/5160, MAE 4190/5190

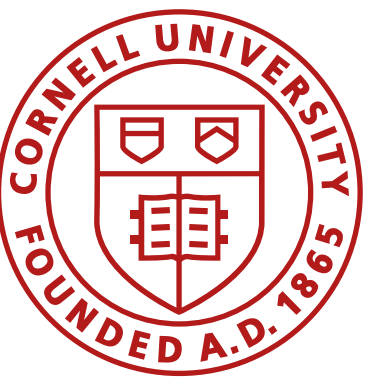
E. Farrell Helbling, 2/26/26

Slides adapted from Prof. Kirstin Petersen

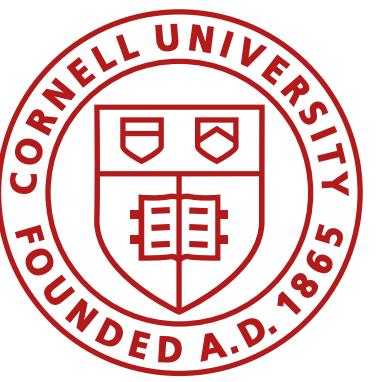


Class Action Items

- Lots of open hours this week and this weekend!
 - Solder help, attachment help, open loop drive help, etc!
- You do not need to put the blue cover back on your car
- Power supply current limits

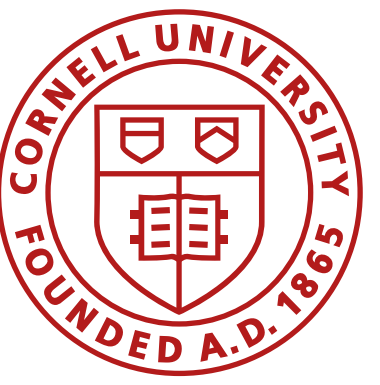


Linear Systems

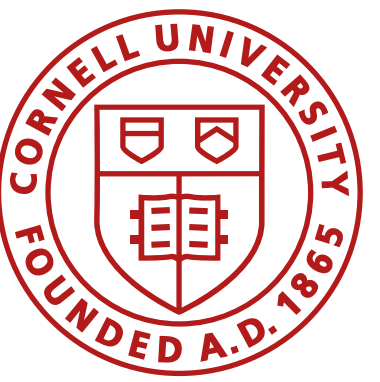


Review

- Linear system: $\dot{x} = Ax$
- Solution: $x(t) = e^{At}x(0)$
- Eigenvectors: $T = [\xi_1 \quad \xi_2 \quad \dots \quad \xi_n]$
- Eigenvalues: $D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$
- `>> [T,D] = eig(A)`
- Linear Transform: $AT = TD$
- Solution: $e^{At} = e^{TDT^{-1}t}$
- Mapping from x to z to x : $x(t) = Te^{Dt}T^{-1}x(0)$
- Stability in continuous time: $\lambda = a + ib$, stable iff $a < 0$
- Discrete time: $x(k+1) = \tilde{A}x(k)$, where $\tilde{A} = e^{A\Delta t}$
- Stability in discrete time: $\tilde{\lambda}^n = R^n e^{in\theta}$, stable iff $R < 1$
- Nonlinear systems: $\dot{x} = f(x)$
- Linearization: $\left. \frac{Df}{Dx} \right|_{\bar{x}}$



Controllability



Controllability

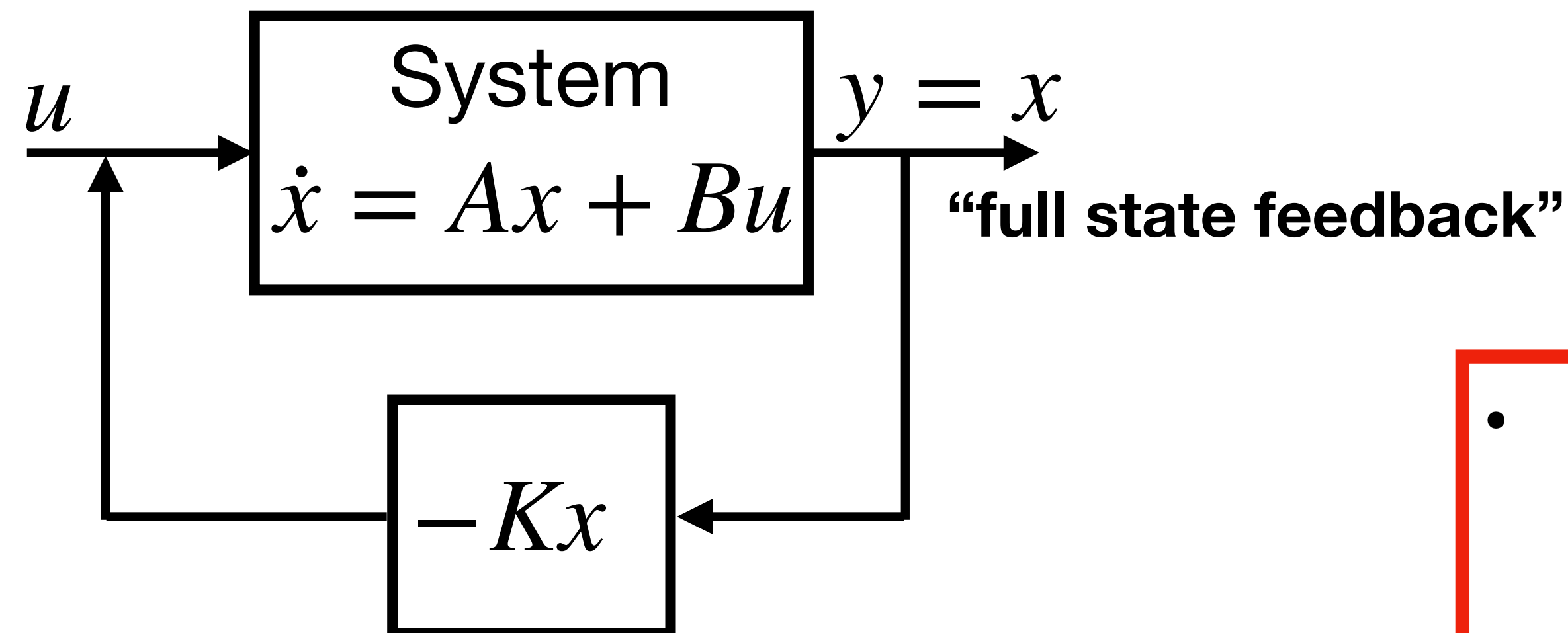
- A system is controllable if you can steer your state x anywhere you want in \mathbb{R}^n

$$\dot{x} = Ax + Bu \quad x \in \mathbb{R}^n$$

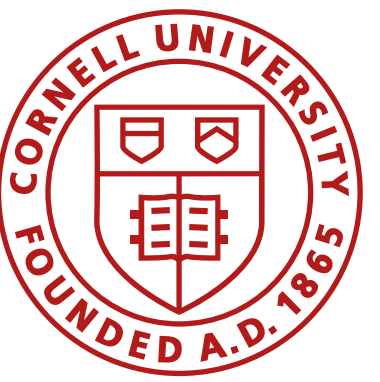
$$\dot{x} = Ax - BKx \quad A \in \mathbb{R}^{n \times n}$$

$$\dot{x} = \underline{(A - BK)}x \quad u \in \mathbb{R}^q$$

New dynamics $B \in \mathbb{R}^{n \times q}$



- Controllability matrix
 - Matlab `>>ctrb(A, B)`
 - $\mathbb{C} = [B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B]$
 - The system is controllable iff $\text{rank}(\mathbb{C}) = n$

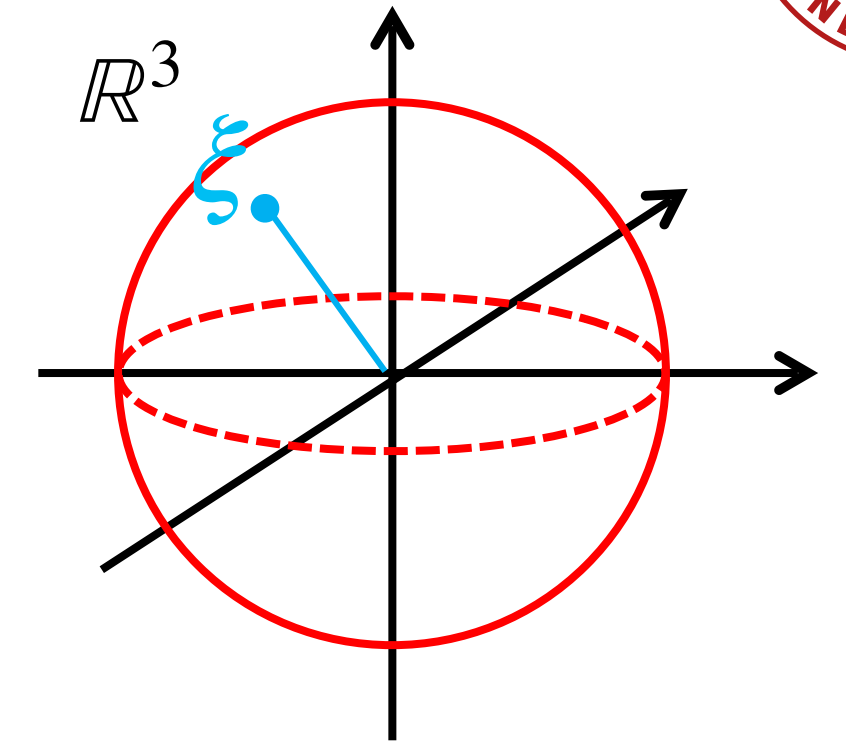


Controllability and Reachability

$$\dot{x} = Ax + Bu \quad x \in \mathbb{R}^n$$

$$\mathbb{C} = [B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B]$$

If the point is reachable,
any point in that direction
is reachable



Equivalences

- The system is controllable
 - iff $\text{rank}(\mathbb{C}) = n$

Reachability

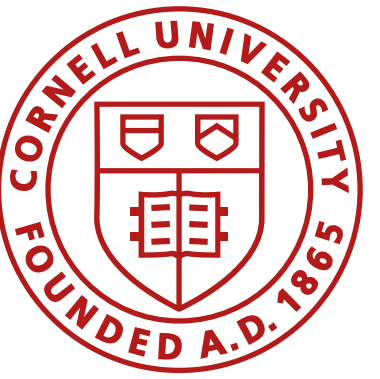
- \mathcal{R}_t : states that are reachable at time t
- $\mathcal{R}_t = \{\xi \in \mathbb{R}^n \text{ for which there is an input } u(t) \text{ that makes } x(t) = \xi\}$

- You can choose K to arbitrarily place the eigenvalues of your closed loop system

- $\dot{x} = (A - BK)x$ `>>K = scipy.signal.place_poles(A, B, poles)`

- You can reach anywhere in \mathbb{R}^n in a finite amount of time and energy

- $\mathcal{R}_t = \mathbb{R}^n$



Controllability Gramians

- $x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$

- Controllability Gramian

- $W_t = \int_0^t e^{A\tau}BB^Te^{A^T\tau}d\tau \quad W_t \in \mathbb{R}^{n \times n}$

- $W_t\xi = \lambda\xi$

- Discrete time

- $W_t \approx CC^T$

$$\dot{x} = Ax + Bu \quad x \in \mathbb{R}^n$$

$$\mathbb{C} = [B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B]$$

$$\gg \text{rank}(\text{ctrb}(A, B))$$

$$\gg [U, S, V] = \text{svd}(\mathbb{C}, \text{'econ'})$$

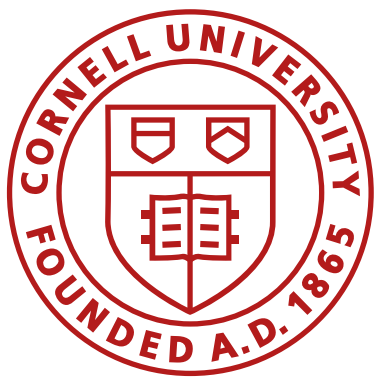
The SVD of A takes the form: $A = U\Sigma V^T$

U = left singular vector

V = right singular vector

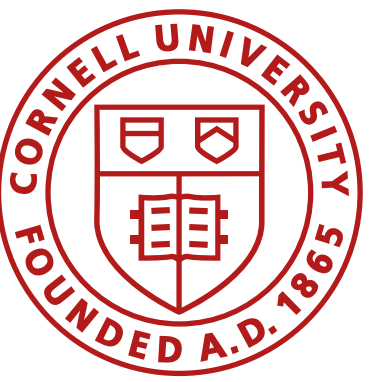
Σ = diagonal matrix of singular values

The eigenvectors with the biggest eigenvalues of the controllability gramian are also the most controllable directions in state space!



Review

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- Eigenvalues: $D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$
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- Nonlinear systems: $\dot{x} = f(x)$
- Linearization: $\left. \frac{Df}{Dx} \right|_{\bar{x}}$
- Controllability: $\dot{x} = (A - BK)x$, controllable if rank = n
- Reachability `>> rank(ctrb(A, B))`
- Controllability Gramian `>> [U, S, V] = svd(C, 'econ')`



Linear Systems – where are we?

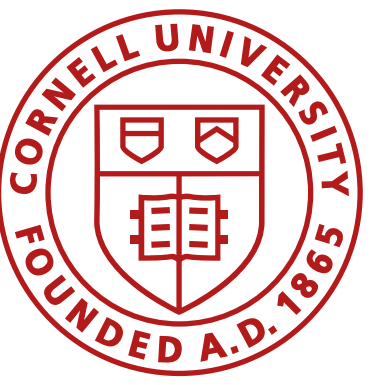
- Linear systems review
- Eigenvectors and eigenvalues
- Stability
- Discrete time systems
- Linearizing nonlinear systems
- Controllability
- LQR control
- Observability

$$\dot{x} = Ax + Bu$$

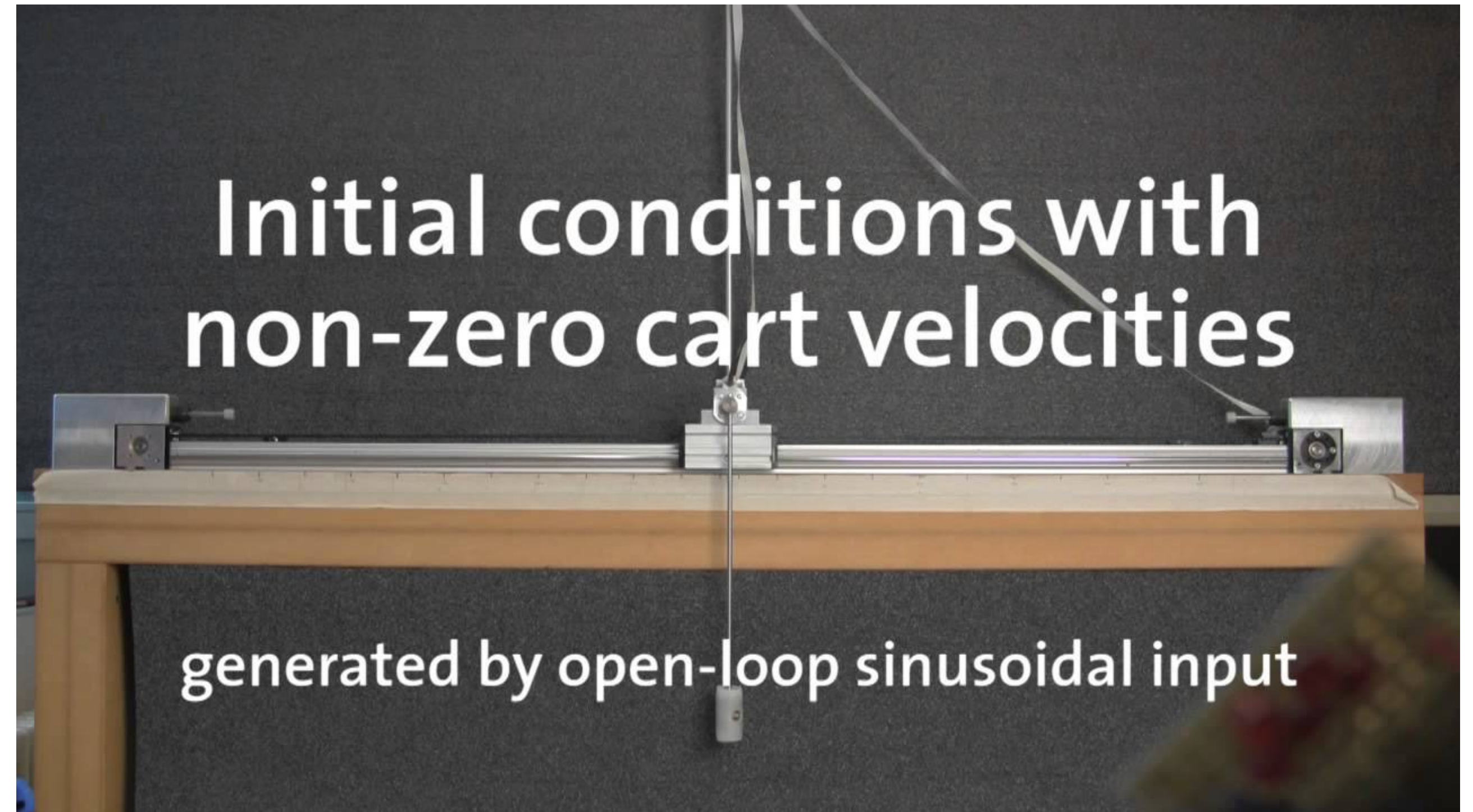
These should look familiar from:

- MATH2940 Linear Algebra
- ECE3250 Signals and Systems
- ECE5210 Theory of Linear Systems
- MAE3260 System Dynamics
- and many others...

Based on “Control Bootcamp”, Steve Brunton, UW
<https://www.youtube.com/watch?v=Pi7l8mMjYVE>



Cart Pole

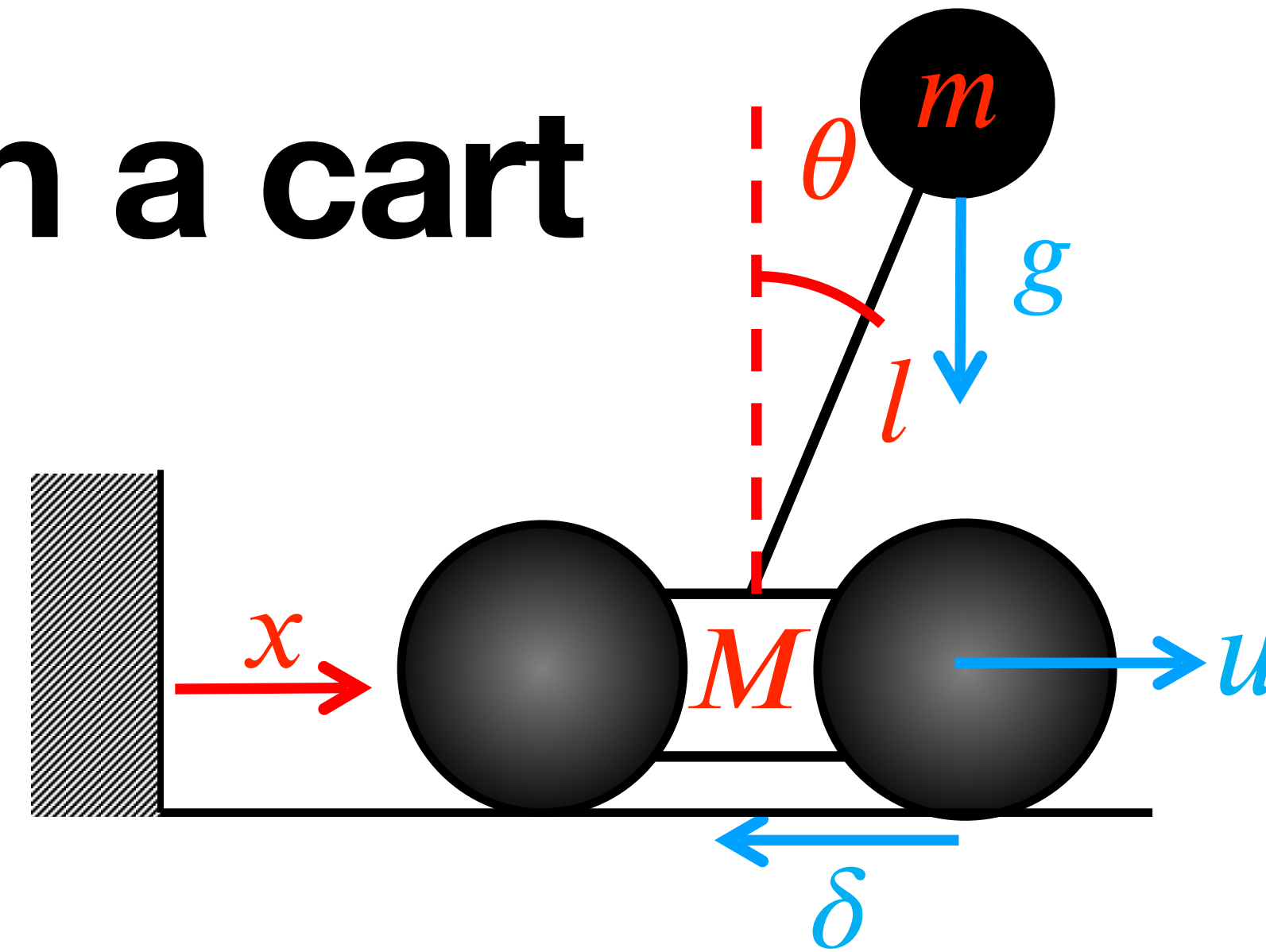


Reist, Tedrake ICRA 2010

Based entirely on Steve Brunton's Controlled Bootcamp Lecture Series

Inverted pendulum on a cart

How do we reason about this system?



Force acting on the cart in the x direction

1. Eqs. of motion



2. State space model

→ 3. Fixed points

→ 4. Jacobian

→ 5. Is it controllable?



6. Add linear control

$$y = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix}$$

$$\bar{y} = \begin{bmatrix} \\ \phantom{\dot{x}} \\ \\ \phantom{\dot{\theta}} \end{bmatrix}$$

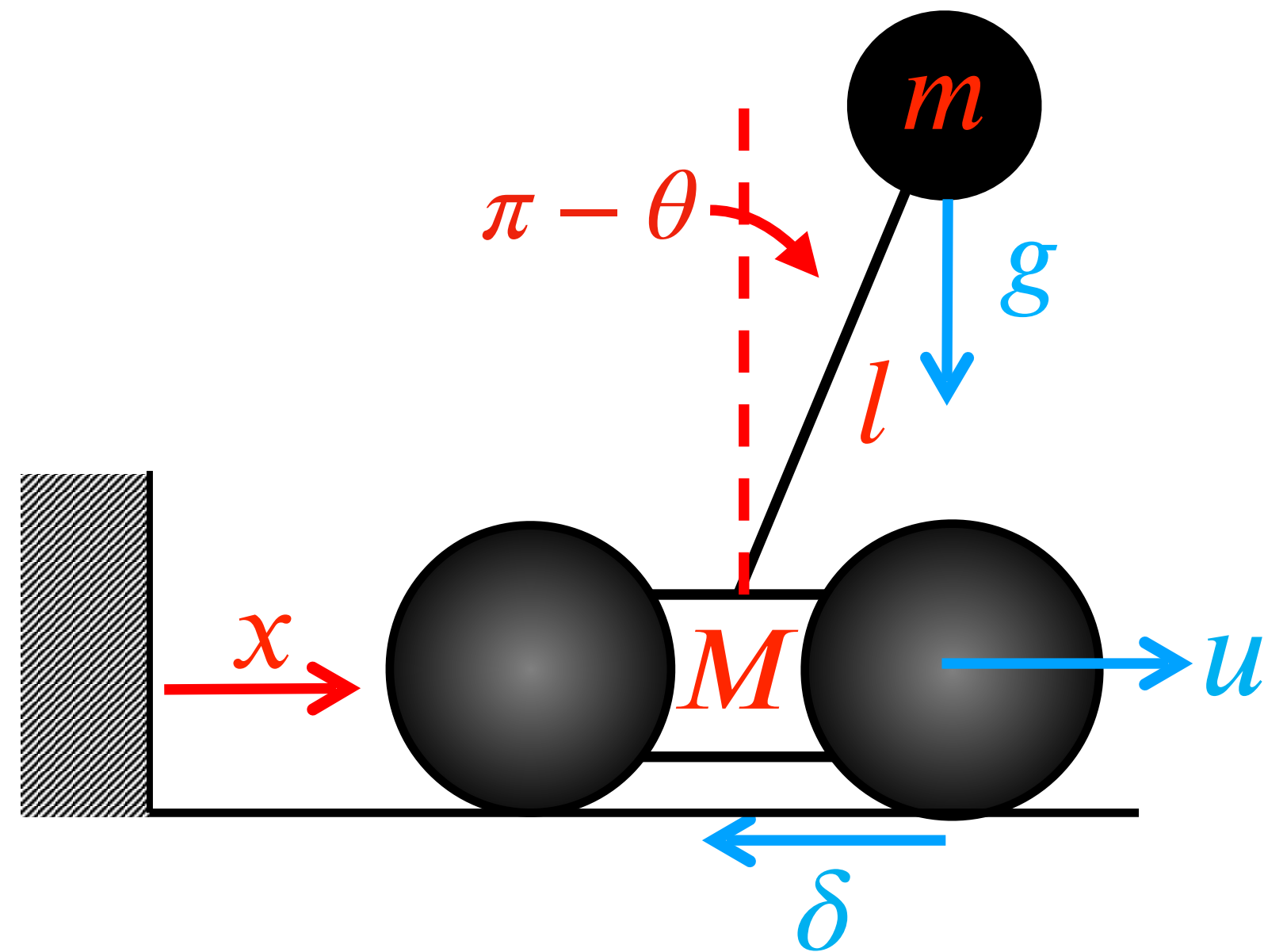
$$\left. \frac{Df}{Dy} \right|_{\bar{y}}$$

$$\dot{y} = Ay + Bu$$

$$\dot{y} = (A - BK)y$$

Inverted pendulum on a cart

Equations of motion



Euler Lagrange Formulation

$$\frac{d}{dt} \left(\frac{\delta L}{\delta \dot{q}_i} \right) - \frac{\delta L}{\delta q_i} = Q_i \quad L = K - P$$

$$P = mgy_m = -mgl \cos(\theta)$$

$$K = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m \dot{x}_m^2 + \frac{1}{2} m \dot{y}_m^2$$

$$K = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m (\dot{x}_m^2 + \dot{y}_m^2)$$

$$x_m = x + l \sin(\theta)$$

$$\dot{x}_m = \dot{x} + l \dot{\theta} \cos(\theta)$$

$$y_m = -l \cos(\theta)$$

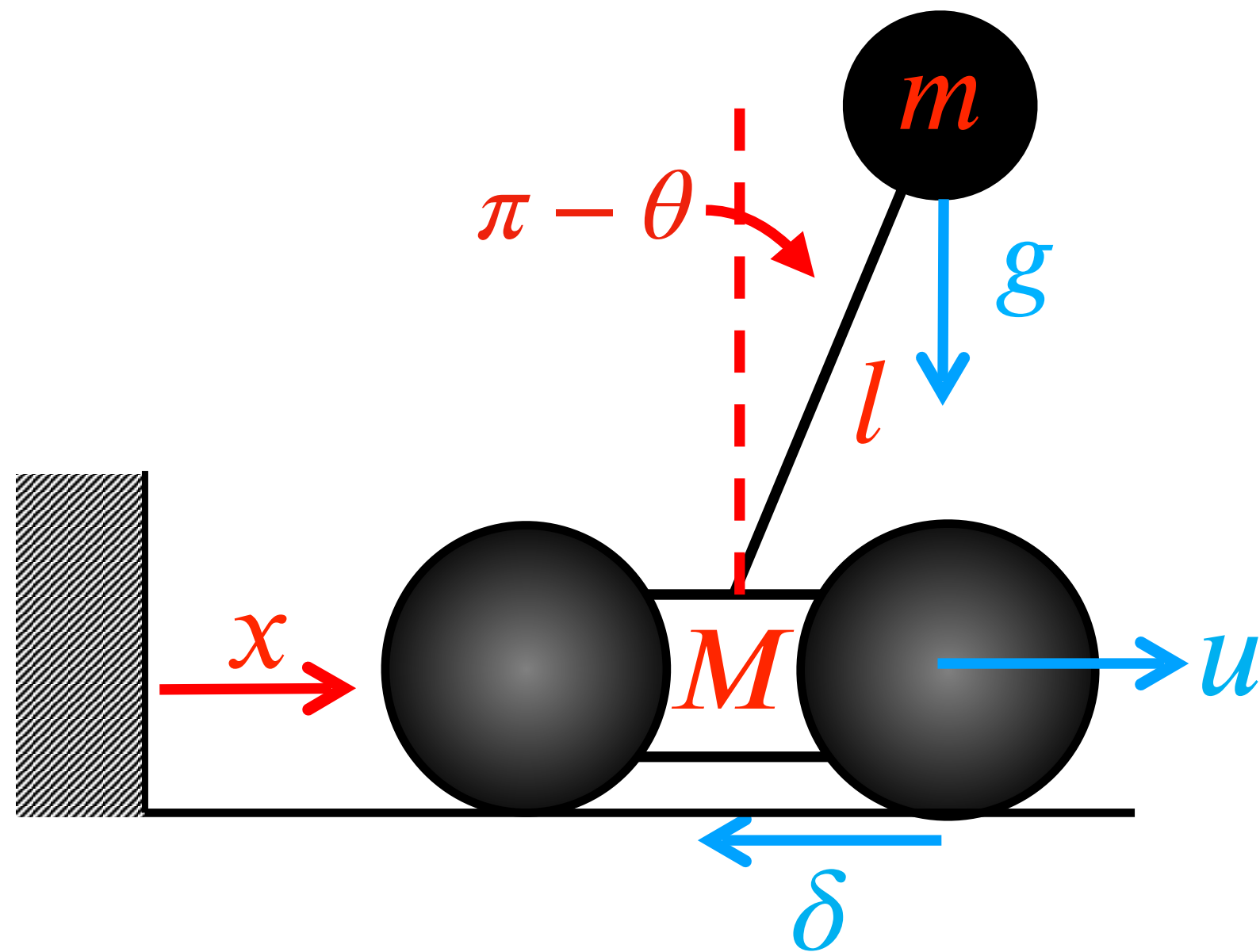
$$\dot{y}_m = l \dot{\theta} \sin(\theta)$$

$$K = \frac{1}{2} M \dot{x}^2 + \frac{1}{2} m (\dot{x}^2 + 2l \dot{\theta} \dot{x} \cos(\theta) + l^2 \dot{\theta}^2 \cos^2(\theta) + l^2 \dot{\theta}^2 \sin^2(\theta))$$

$$K = \frac{1}{2} (M + m) \dot{x}^2 + \frac{1}{2} ml^2 \dot{\theta}^2 + ml \dot{\theta} \dot{x} \cos(\theta)$$

Inverted pendulum on a cart

Equations of motion in x



Euler Lagrange Formulation

$$\frac{d}{dt} \left(\frac{\delta L}{\delta \dot{q}_i} \right) - \frac{\delta L}{\delta q_i} = Q_i$$

$$L = \frac{1}{2}(M + m)\dot{x}^2 + \frac{1}{2}ml^2\dot{\theta}^2 + ml\dot{x}\cos(\theta) + mgl\cos(\theta)$$

$$q_i = x, \quad Q_i = F - \delta\dot{x}$$

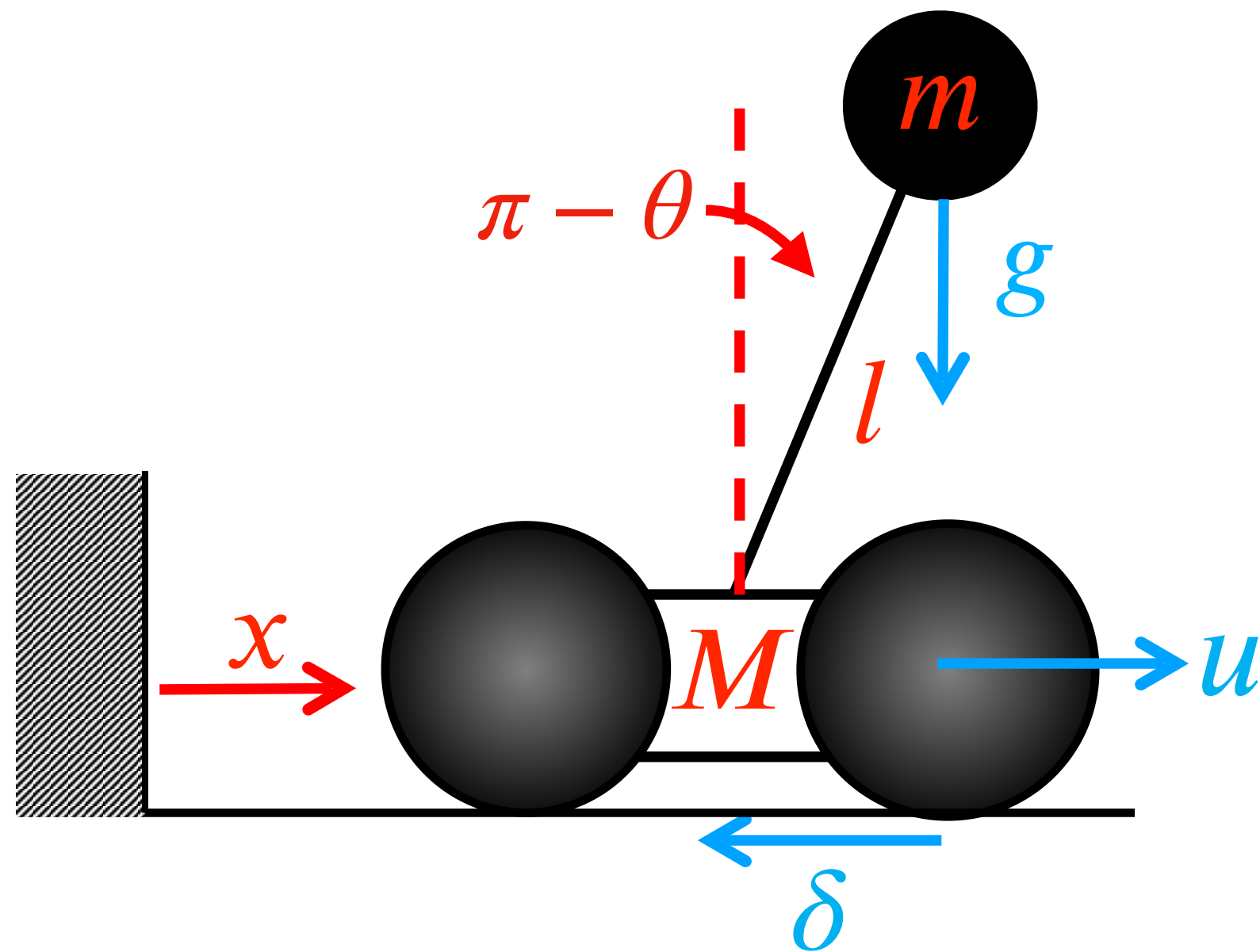
$$\frac{\delta L}{\delta \dot{x}} = (M + m)\dot{x} + ml\dot{\theta}\cos(\theta) \quad \frac{\delta L}{\delta x} = 0$$

$$\frac{d}{dt} \left(\frac{\delta L}{\delta \dot{x}} \right) = (M + m)\ddot{x} + ml\ddot{\theta}\cos(\theta) - ml\dot{\theta}^2\sin(\theta)$$

$$(M + m)\ddot{x} + ml\ddot{\theta}\cos(\theta) - ml\dot{\theta}^2\sin(\theta) = F - \delta\dot{x}$$

Inverted pendulum on a cart

Equations of motion in θ



Euler Lagrange Formulation

$$\frac{d}{dt} \left(\frac{\delta L}{\delta \dot{q}_i} \right) - \frac{\delta L}{\delta q_i} = Q_i$$

$$L = \frac{1}{2}(M + m)\dot{x}^2 + \frac{1}{2}ml^2\dot{\theta}^2 + ml\dot{x}\cos(\theta) + mgl\cos(\theta)$$

$$q_i = \theta, \quad Q_i = 0$$

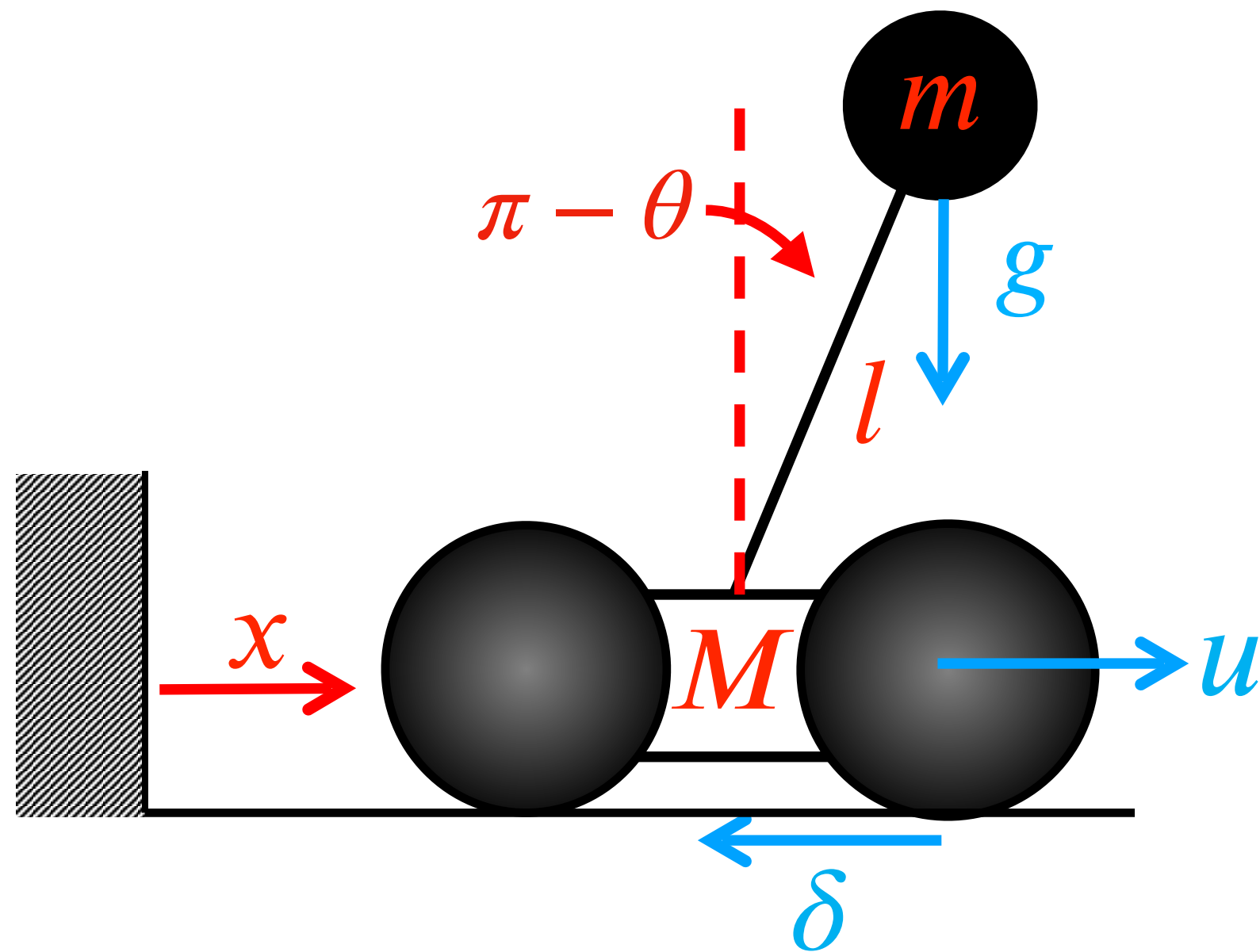
$$\frac{\delta L}{\delta \dot{\theta}} = ml^2\dot{\theta} + ml\dot{x}\cos(\theta) \quad \frac{\delta L}{\delta \theta} = -ml\dot{x}\sin(\theta) - mgl\sin(\theta)$$

$$\frac{d}{dt} \left(\frac{\delta L}{\delta \dot{\theta}} \right) = ml^2\ddot{\theta} + ml\ddot{x}\cos(\theta) - ml\dot{x}\dot{\theta}\sin(\theta)$$

$$ml^2\ddot{\theta} + ml\ddot{x}\cos(\theta) + mgl\sin(\theta) = 0$$

Inverted pendulum on a cart

Equations of motion



$$(M + m)\ddot{x} + ml\ddot{\theta} \cos(\theta) - ml\dot{\theta}^2 \sin(\theta) = F - \delta\dot{x}$$

$$ml^2\ddot{\theta} + ml\ddot{x} \cos(\theta) + mgl \sin(\theta) = 0$$

$$\begin{bmatrix} (M + m) & ml \cos(\theta) \\ ml \cos(\theta) & ml^2 \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} F + ml\dot{\theta}^2 \sin(\theta) - \delta\dot{x} \\ -mgl \sin(\theta) \end{bmatrix}$$

$$\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{(ml^2)(F + ml\dot{\theta}^2 \sin(\theta) - \delta\dot{x} + mg \cos(\theta)\sin(\theta))}{\Delta} \\ \frac{(-ml \cos(\theta))(F + ml\dot{\theta}^2 \sin(\theta) - \delta\dot{x}) - (M + m)mgl \sin(\theta)}{\Delta} \end{bmatrix}$$

$$\det = \Delta = ml^2(M + m(1 - \cos^2(\theta)))$$

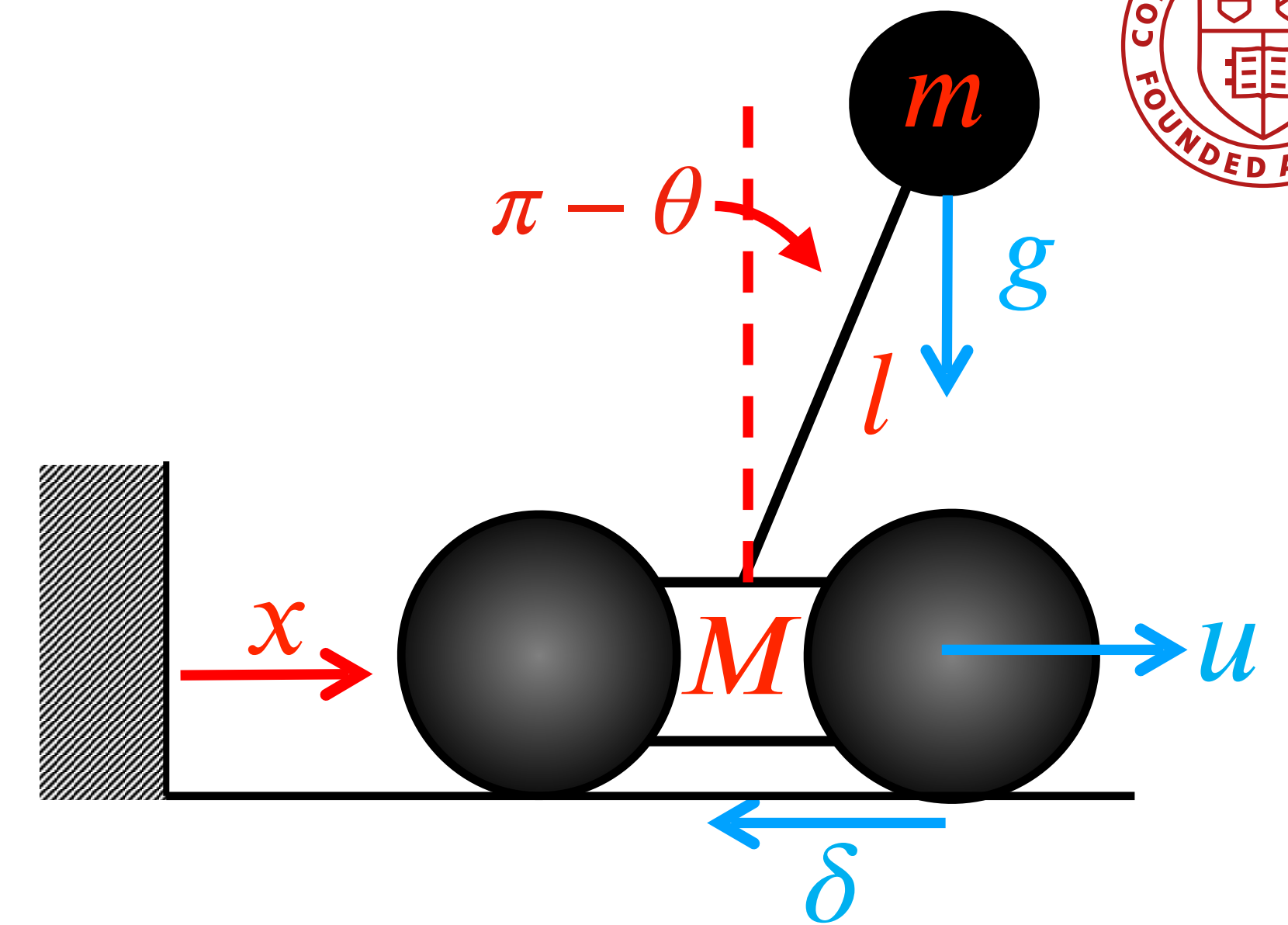
Inverted pendulum on a cart

Linearize the nonlinear system

$$\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \frac{(ml^2)(F + ml\dot{\theta}^2 \sin(\theta) - \delta\dot{x} + mg \cos(\theta)\sin(\theta))}{\Delta} \\ \frac{(-ml \cos(\theta))(F + ml\dot{\theta}^2 \sin(\theta) - \delta\dot{x}) - (M + m)mgl \sin(\theta)}{\Delta} \end{bmatrix}$$

$$\Delta = ml^2(M + m(1 - \cos^2(\theta)))$$

$$\frac{d}{dt} \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{x} \\ \frac{(ml^2)(F + ml\dot{\theta}^2 \sin(\theta) - \delta\dot{x} + mg \cos(\theta)\sin(\theta))}{\Delta} \\ \dot{\theta} \\ \frac{(-ml \cos(\theta))(F + ml\dot{\theta}^2 \sin(\theta) - \delta\dot{x}) - (M + m)mgl \sin(\theta)}{\Delta} \end{bmatrix}$$



Linearize about:

$$x = \text{free}, \dot{x} = 0, \theta = \{0, \pi\}, \dot{\theta} = 0$$

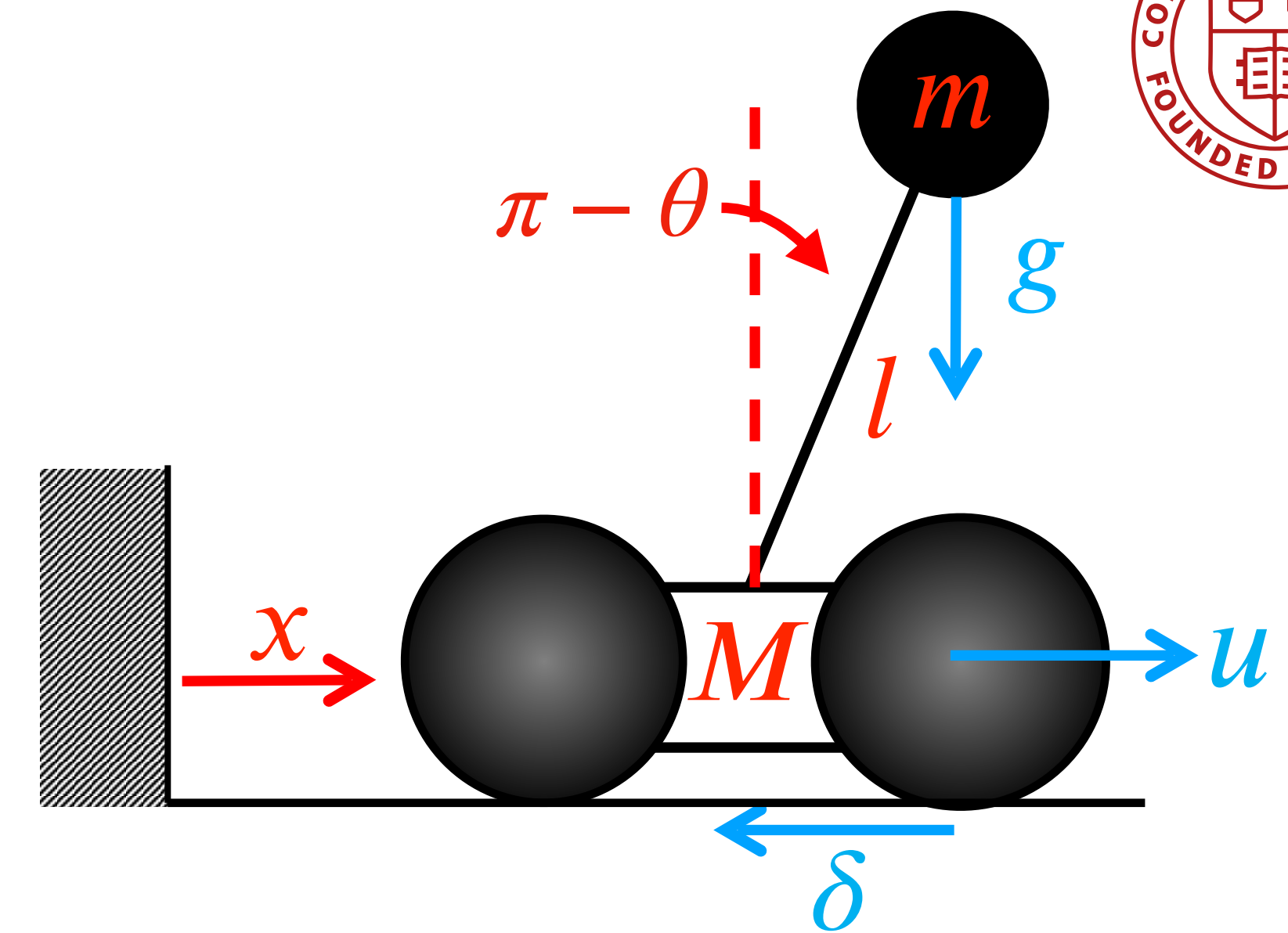
$$\frac{d}{dt} \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{\delta}{M} & \frac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{\text{up} \delta}{ML} & \frac{\text{up}(m+M)g}{ML} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{\text{up}}{ML} \end{bmatrix} F$$

where $\text{up} = 1$ at $\theta = \pi$ and $\text{up} = -1$ at $\theta = 0$

Inverted pendulum on a cart

Eigenvalues, Stability, Controllability

$$\frac{d}{dt} \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{\delta}{M} & \frac{mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{\text{up } \delta}{ML} & \frac{\text{up}(m+M)g}{ML} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{\text{up}}{ML} \end{bmatrix} F$$



Let's go to Matlab!

- Check nonlinear equations
- Run open-loop simulation
- Check for stability, controllability

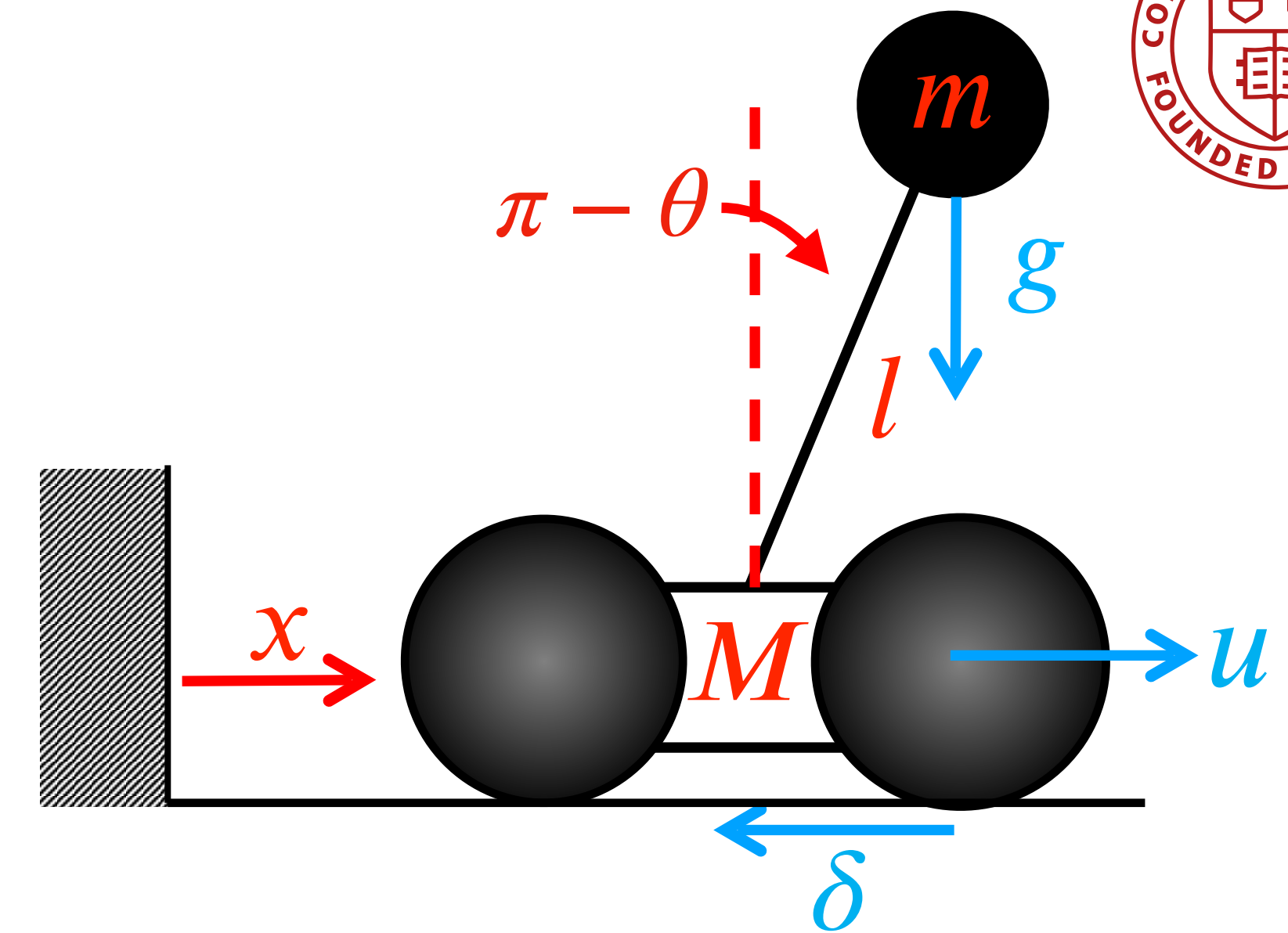
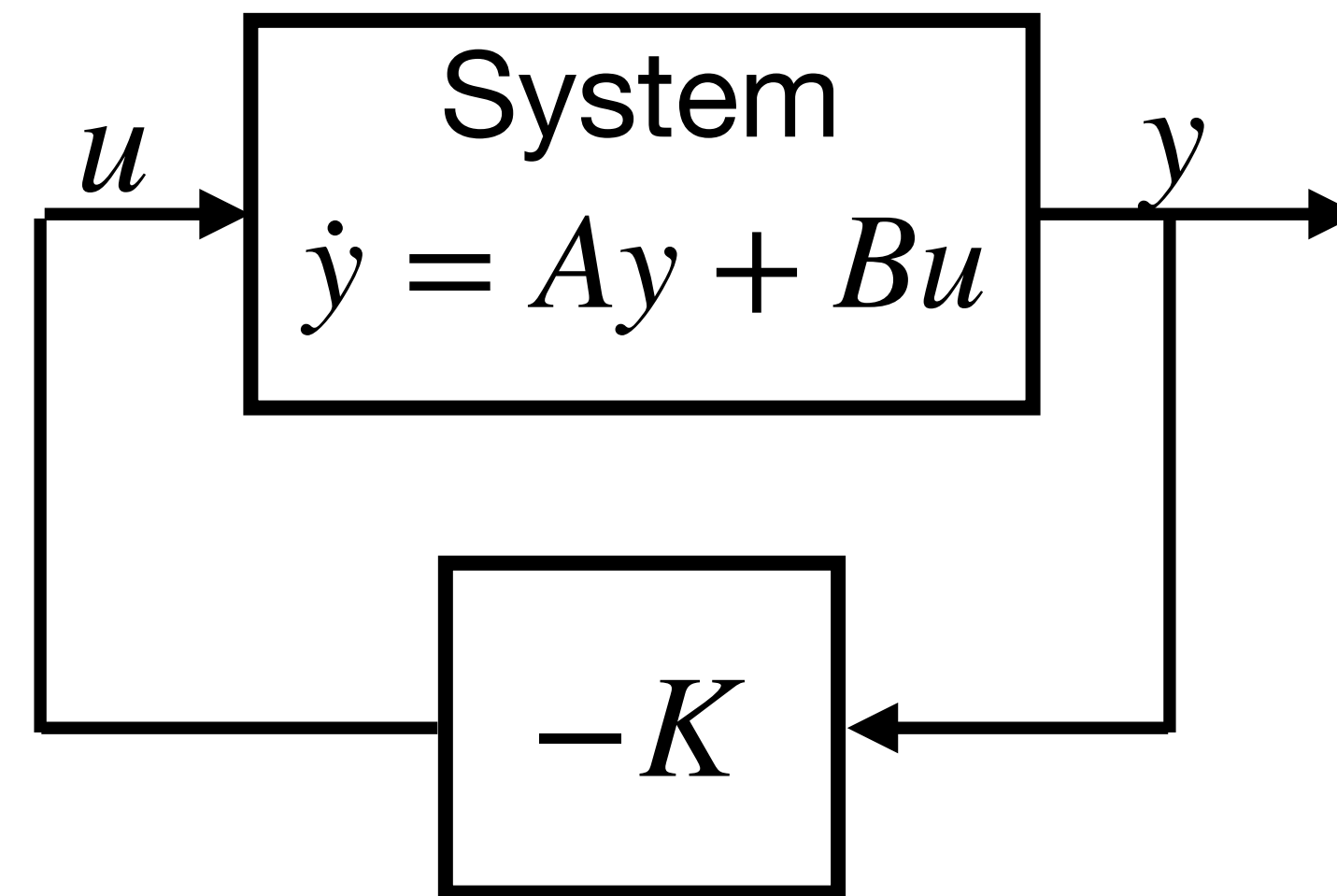
Inverted pendulum on a cart

Control Law

$$\dot{y} = Ay + Bu$$

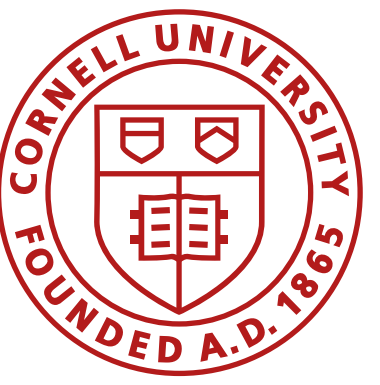
$$u = -Ky$$

$$\dot{y} = (A - BK)y$$



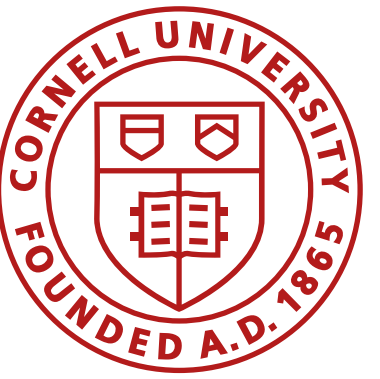
Let's go to Matlab!

- Pole placement.
- Define poles `>>eigs = [-1 -1.2 -1.3 -1.4];`
- K-matrix `>>K = place(A,B,eigs)`



Pole Placement

- Python
 - `K = scipy.signal.place_poles(A, B, poles)`¹
- Barely stable eigenvalues: not enough control authority
- More negative eigenvalues: faster response, less robust system



Linear Quadratic Regulator

- What are the optimal eigenvalues for our system?
 - Tradeoff performance and control effort

- Define cost function: $\int_0^{\infty} (x^T Q x + u^T R u) dt$

- $Q = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 10 & \\ & & & 100 \end{bmatrix}$ cost of my state being away from setpoint

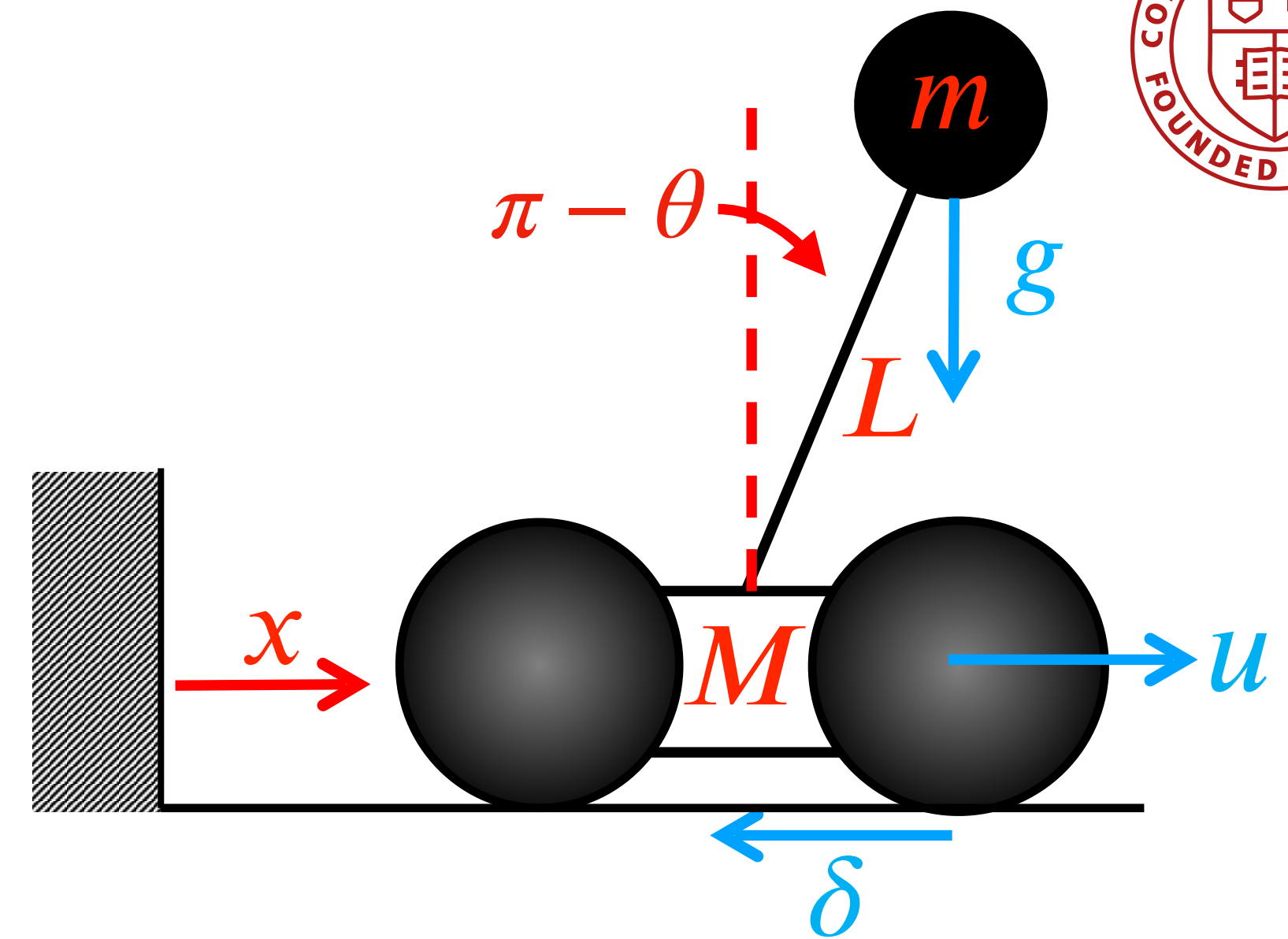
- $R = 0.001$ cost of input energy
- Solved using the Ricatti Equation (compute expensive $O(n^3)$)
- Matlab `>>lqr (A, B, Q, R)`

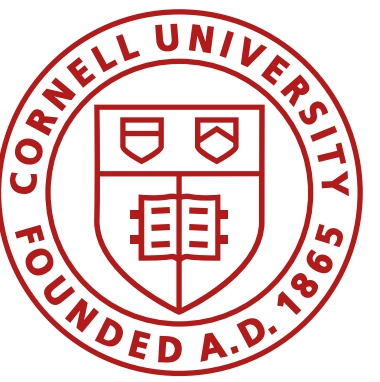
Inverted pendulum on a cart

The controller works!

Caveats:

- In simulation
- Practical issues:
 - Imperfect models
 - Nonlinear parts: deadband, saturation, etc.
 - Partial state feedback





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- Linearization: $\left. \frac{Df}{Dx} \right|_{\bar{x}}$

- Controllability: $\dot{x} = (A - BK)x$ `>> rank(ctrb(A, B))`

- Reachability

- Controllability Gramian

- Pole Placement `>> place(A, B, poles)`

- Optimal Control (LQR) `>> LQR(A, B, Q, R)`