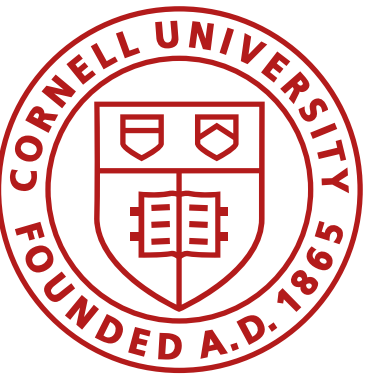


Linear Systems Recap

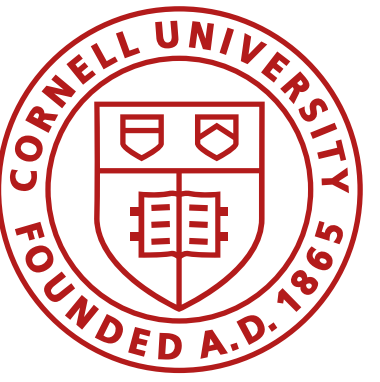
Fast Robots, ECE4160/5160, MAE 4190/5190

E. Farrell Helbling, 2/19/26



Class Action Items

- Lab 3 is due Feb 24-25, if you need to use a slip week, please send us a private message on Ed.
- Lab 4 starts next week, if you want to get a head start during open hours, we will discuss the lab in class today and the website is already posted.
 - Lab 4 has another soldering component, so think about how you want to connect things ahead of time!
 - At the end of Lab 4, you will have a fully-integrated RC car.
 - Good example from last year: <https://boltstrike.github.io/pages/lab4.html>

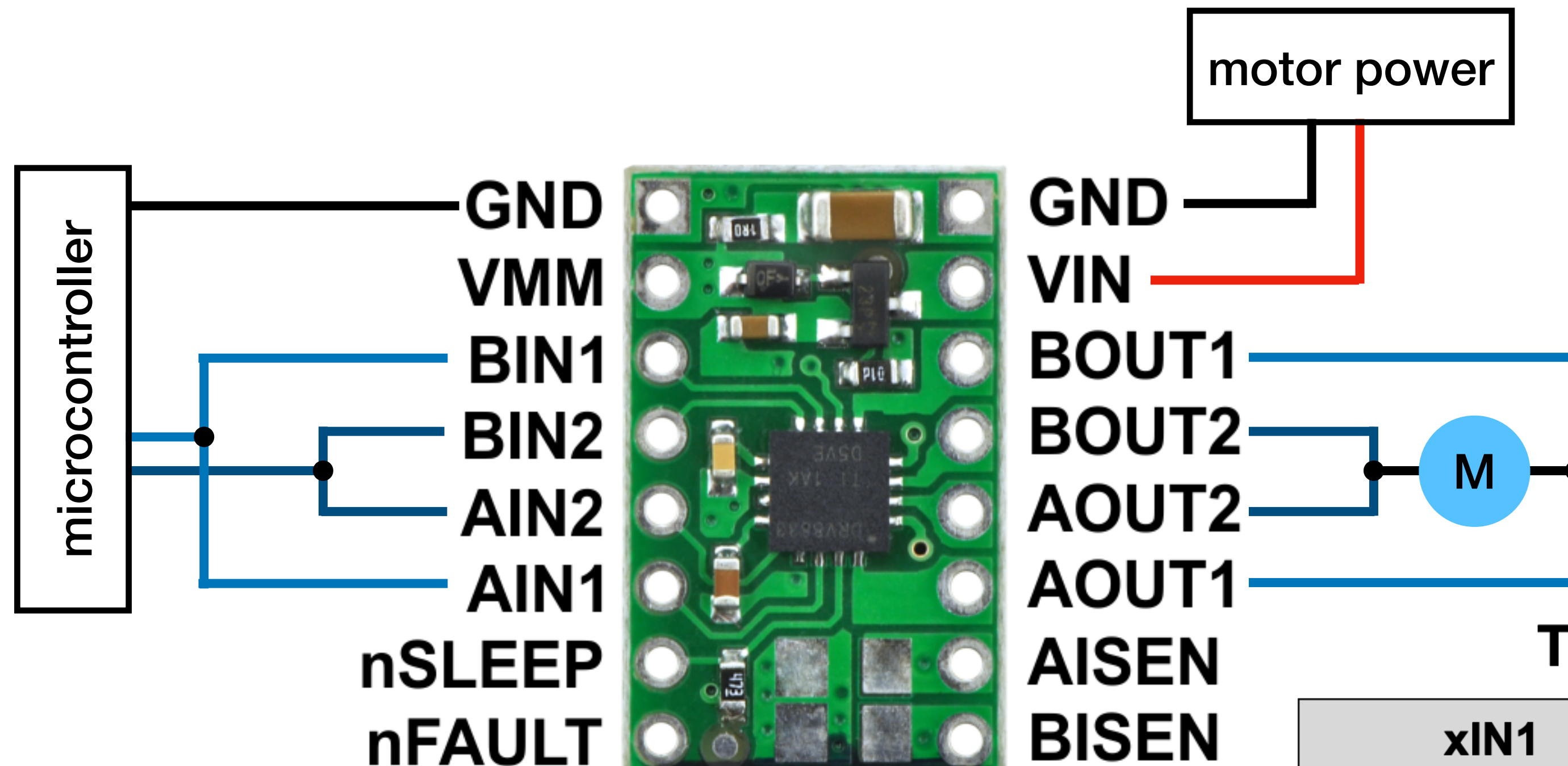


Lab 4 Open loop control

<https://boltstrike.github.io/pages/lab4.html>

Brushed DC motor controllers

Parallel-coupled motor controller

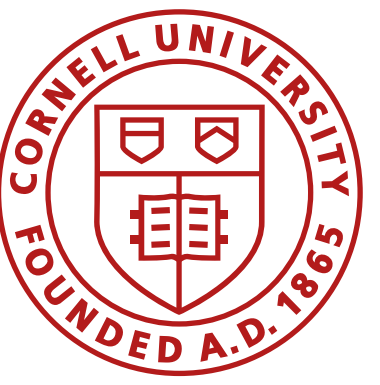


Fast decay: “coasting”

Slow decay: “braking”

Table 2. PWM Control of Motor Speed

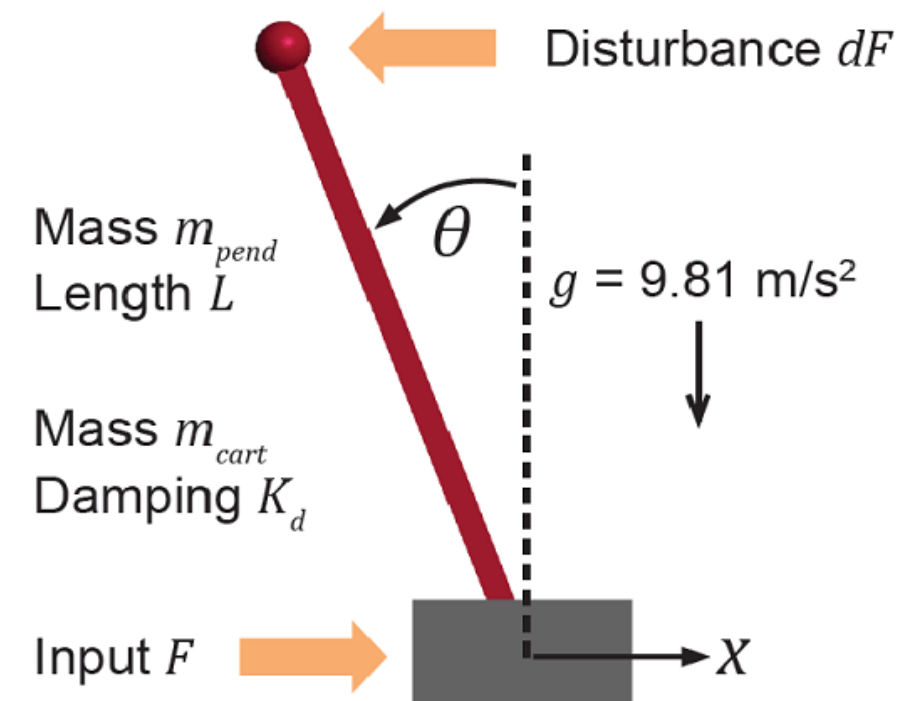
xIN1	xIN2	FUNCTION
PWM	0	Forward PWM, fast decay
1	PWM	Forward PWM, slow decay
0	PWM	Reverse PWM, fast decay
PWM	1	Reverse PWM, slow decay



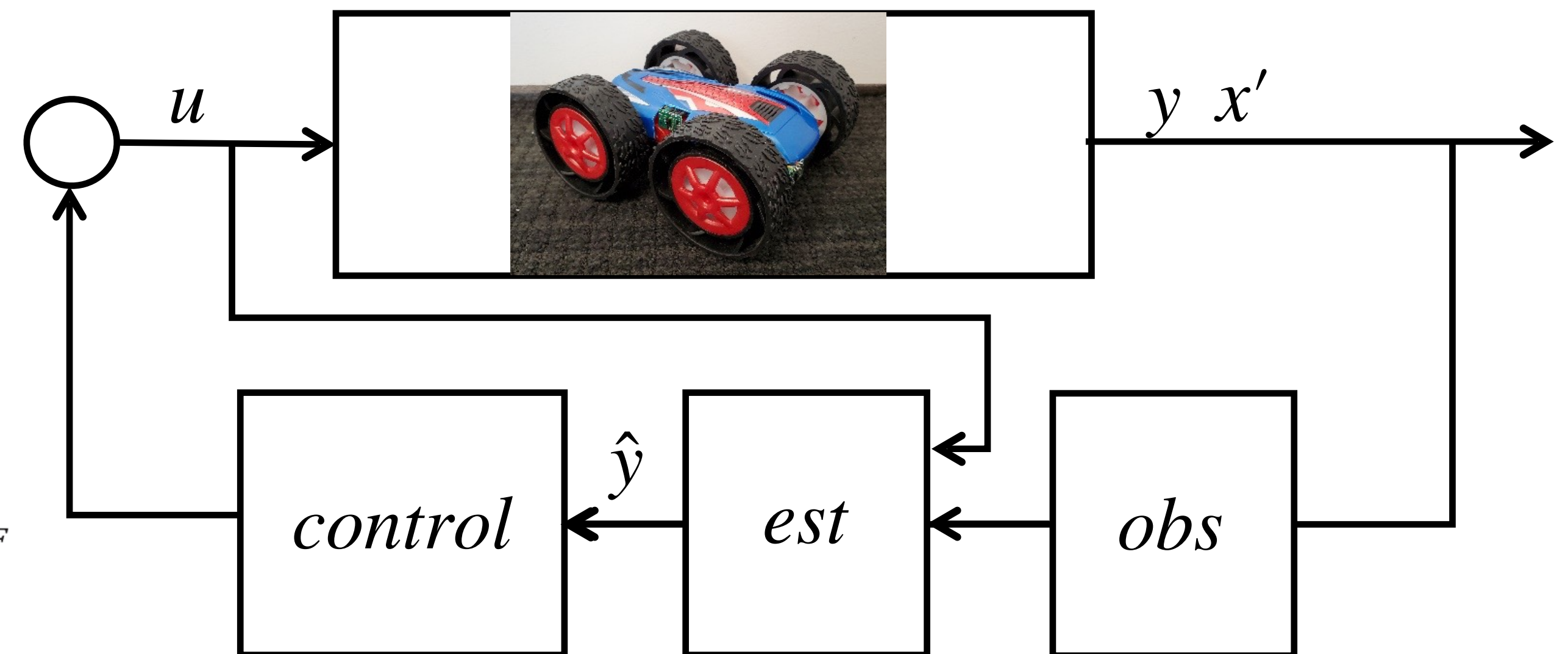
Linear Systems Recap

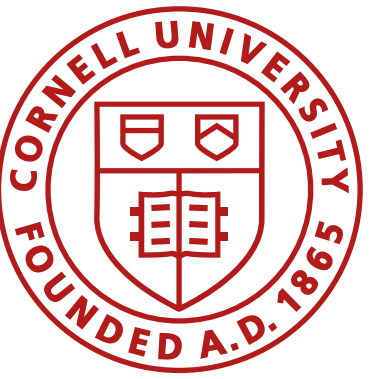
Linear Systems

- Linear systems review
- Eigenvectors and eigenvalues
- Stability
- Discrete time systems
- Linearizing nonlinear systems
- Controllability
- Observability



$$\dot{x} = Ax + Bu$$





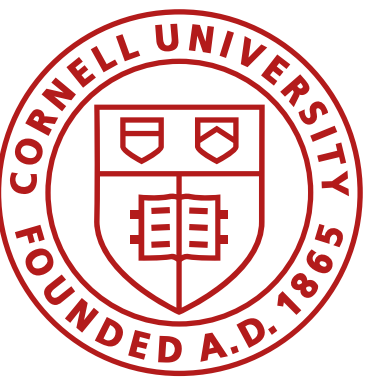
Linear Systems

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$$\dot{x} = Ax + Bu$$

These should look familiar from:

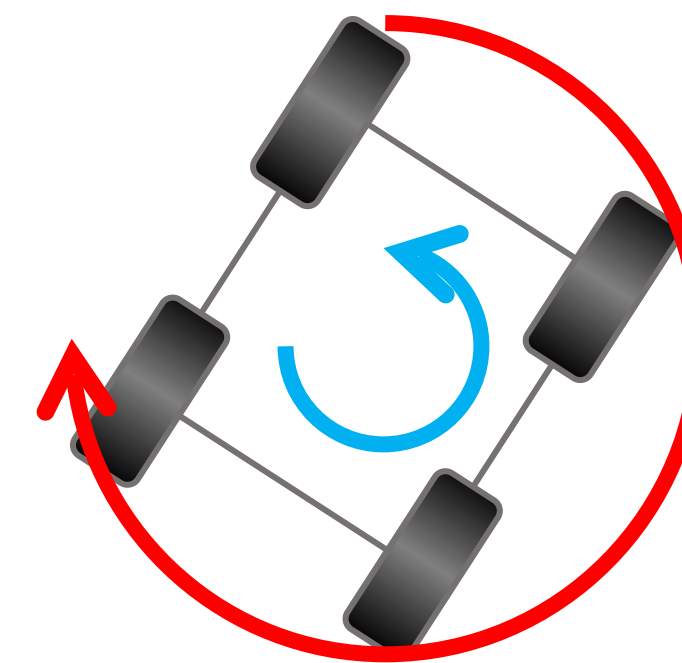
- MATH2940 Linear Algebra
- ECE3250 Signals and Systems
- ECE5210 Theory of Linear Systems
- MAE3260 System Dynamics
- and many others...



Linear Systems

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$$\dot{x} = Ax + Bu$$

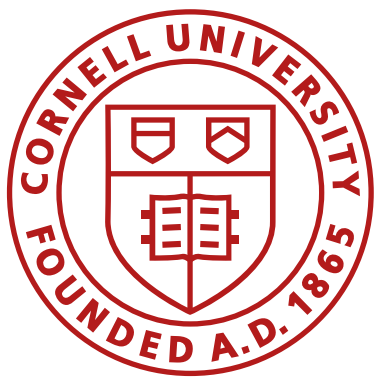


- 1st order system:

$$\begin{bmatrix} \dot{\theta} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & \frac{-c}{I} \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{I} \end{bmatrix} u$$

- 2nd order system:

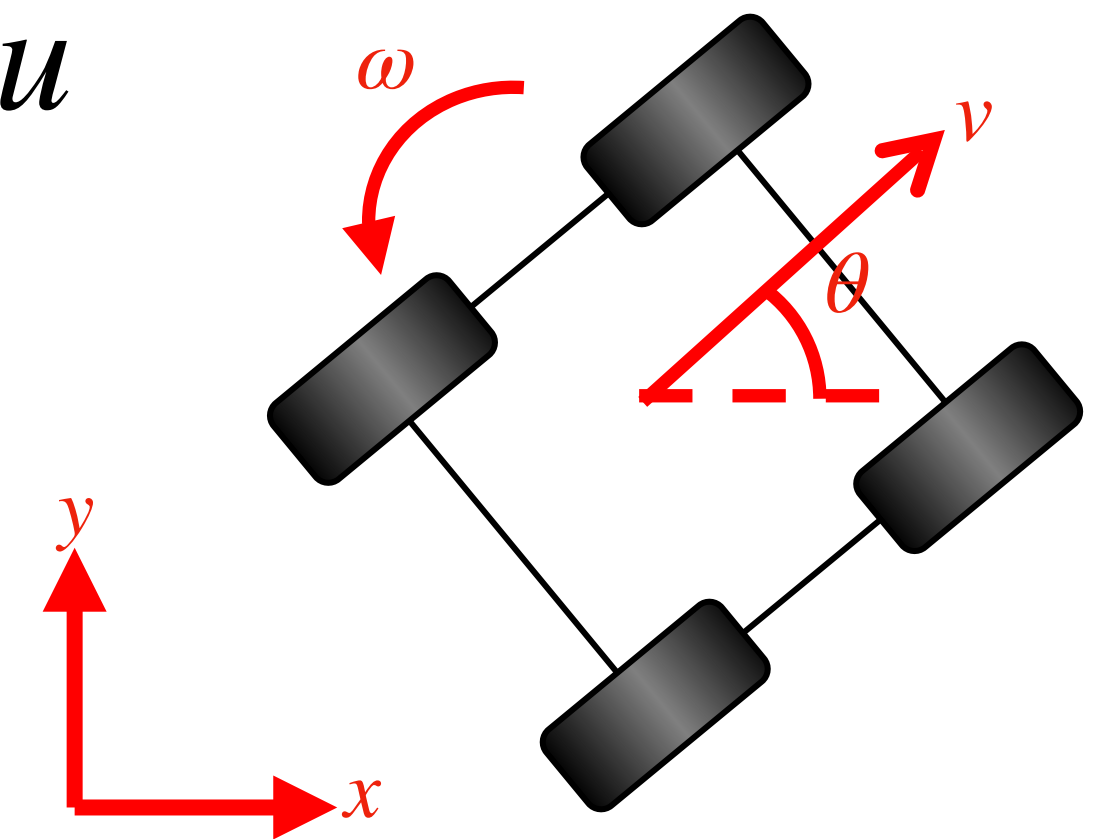
$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \text{const} & \frac{-c}{I} \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{I} \end{bmatrix} u$$



Linear Systems

- Linear systems review
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-
- Controllability
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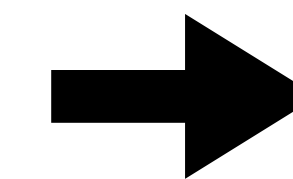
$$\dot{x} = Ax + Bu$$



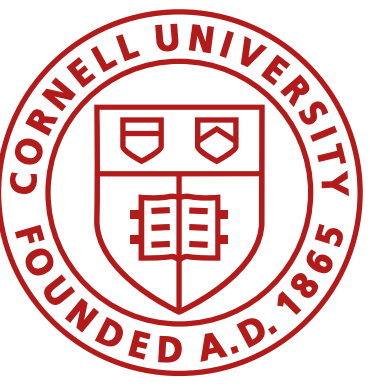
$$\dot{x} = \cos(\theta)v$$

$$\dot{y} = \sin(\theta)v$$

$$\dot{\theta} = \omega$$



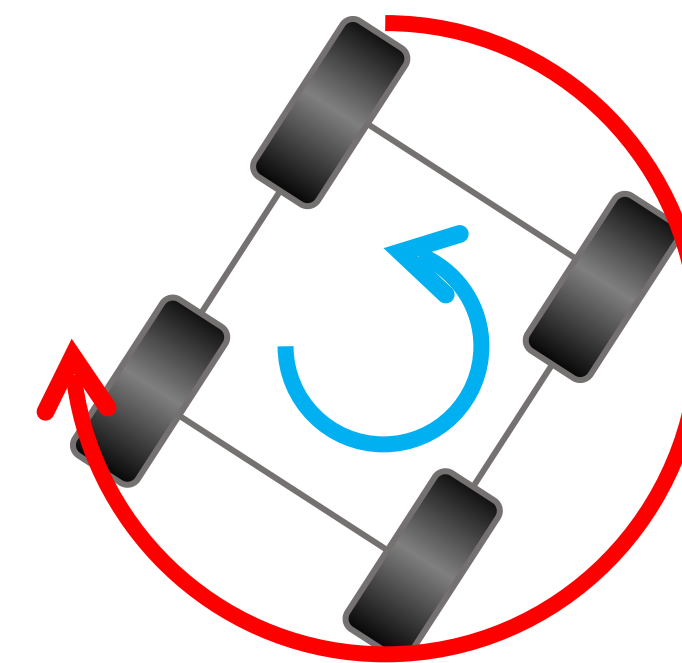
$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos(\theta) & 0 \\ \sin(\theta) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}$$



Linear Systems

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$$\dot{x} = Ax + Bu$$

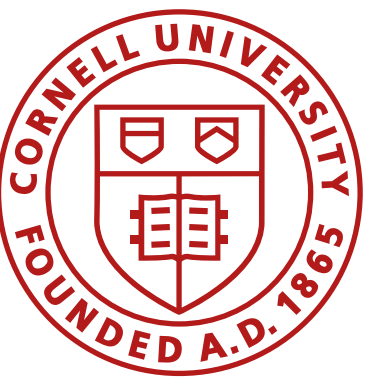


- 1st order system:

$$\begin{bmatrix} \dot{\theta} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & \frac{-c}{I} \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{I} \end{bmatrix} u$$

- 2nd order system:

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \text{const} & \frac{-c}{I} \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{I} \end{bmatrix} u$$



Linear Systems

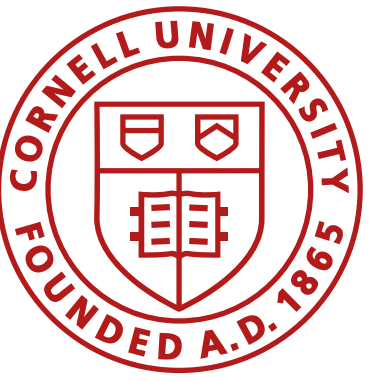
- Linear system $\dot{x} = Ax \quad x \in \mathbb{R}^n \quad A \in \mathbb{R}^{n \times n}$

- Basic solution $x(t) = e^{At}x(0)$

Aside: $\frac{dx}{dt} = kx \leftrightarrow \frac{dx}{x} = kdt \leftrightarrow \ln(|x|) = kt + c$
 $|x| = e^{kt} + e^c \leftrightarrow x = \pm ce^{kt}$

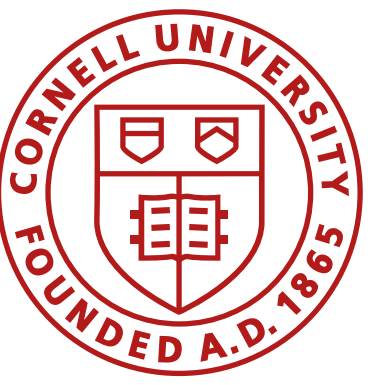
- Taylor series expansion $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots$$

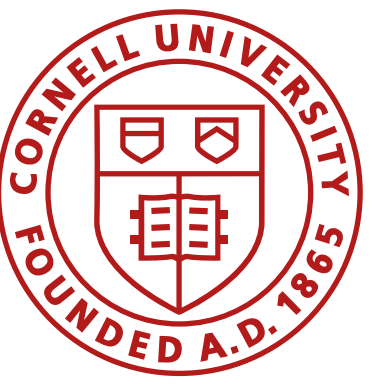


Linear Systems

- Linear system $\dot{x} = Ax$
- Basic solution $x(t) = e^{At}x(0)$
- Map the system to eigenvector coordinates to make computation easier
 - Apply a linear transform: $z = Tx \iff x = T^{-1}z$
 - Substitute into the original equation: $T^{-1}\dot{z} = AT^{-1}z \iff \dot{z} = TAT^{-1}z$
 - Pick the matrix, T , such that TAT^{-1} becomes simpler than A



Eigenvectors and Eigenvalues



Eigenvectors and Eigenvalues

- Eigenvectors, ξ , of A $A\xi = \lambda\xi$
- Matrix of eigenvectors, T

$$T = [\xi_1 \quad \xi_2 \quad \dots \quad \xi_n]$$

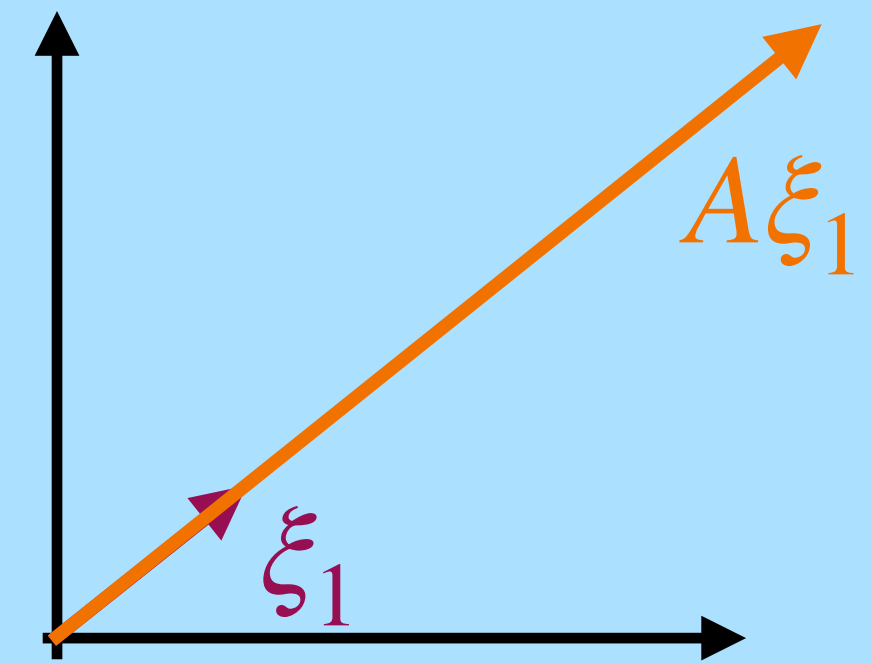
- Diagonal matrix of eigenvalues, D

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$$AT = TD$$

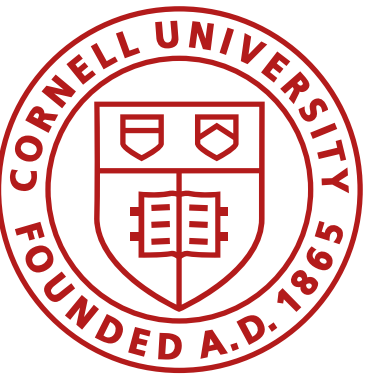
$$A = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$$

$$\xi_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$



$$\begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 8 \end{bmatrix} = 4 \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\lambda_1 = 4$$



z-coordinates

$$\dot{x} = Ax \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$x(t) = e^{At}x(0)$$

$$x = Tz$$

$$\dot{x} = T\dot{z} = Ax$$

$$T\dot{z} = ATz$$

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots$$

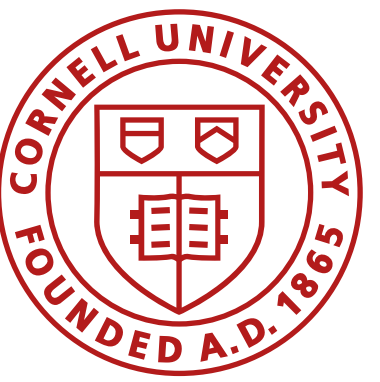
$$\dot{z} = T^{-1}ATz$$

$$AT = TD \quad \Leftrightarrow \quad T^{-1}AT = D$$

$$\dot{z} = Dz$$

MATLAB has a handy function: $[T, D] = \text{eig}(A)$;

By mapping our system to eigenvector coordinates, the dynamics become diagonal (very simple!)



z-coordinates

$$\dot{x} = Ax = T\dot{z}$$

$$x(t) = e^{At}x(0)$$

$$T^{-1}AT = D$$

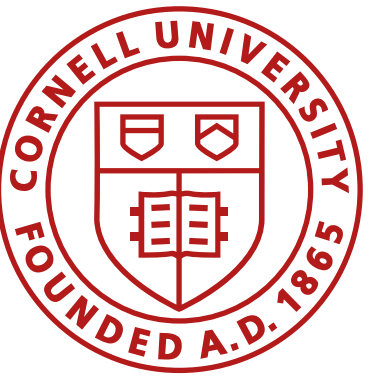
$$\dot{z} = Dz$$

$$\frac{d}{dt} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$$

$$z_1(t) = e^{\lambda_1 t} z_1(0) \quad \dots \quad z_n(t) = e^{\lambda_n t} z_n(0)$$

$$z(t) = e^{Dt} z(0) = \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix} z_0$$

Much simpler to think about the system in eigen coordinates!



Let's get back to x-coordinates

$$\dot{x} = Ax = T\dot{z}$$

$$x(t) = e^{At}x(0)$$

$$T^{-1}AT = D$$

$$\dot{z} = Dz$$

$$A^n = TD^nT^{-1}$$

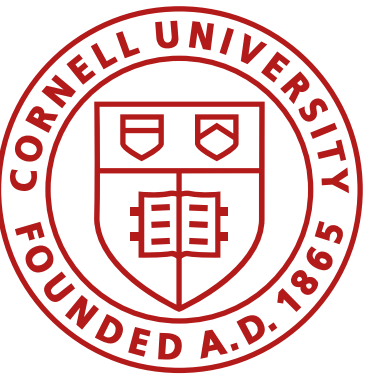
$$x(t) = e^{At}x(0)$$

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots$$

$$e^{At} = e^{TDT^{-1}t}$$

$$e^{At} = I + TDT^{-1}t + \frac{(TDT^{-1}TDT^{-1})}{I} \frac{t^2}{2!} + \dots$$

$$\frac{(TD^2T^{-1})}{I}$$



Let's get back to x-coordinates

$$\dot{x} = Ax = T\dot{z}$$

$$x(t) = e^{At}x(0)$$

$$T^{-1}AT = D$$

$$\dot{z} = Dz$$

$$A^n = TD^nT^{-1}$$

$$x(t) = e^{At}x(0)$$

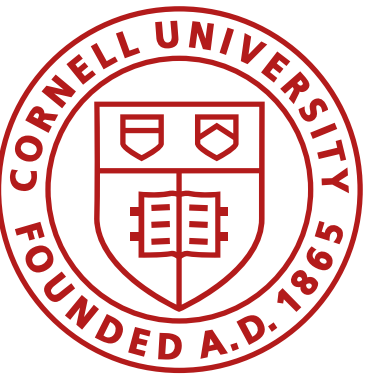
$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots$$

$$e^{At} = e^{TDT^{-1}t}$$

$$e^{At} = I + TDT^{-1}t + \left(TDT^{-1}TDT^{-1}\right) \frac{t^2}{2!} + \dots$$

$$e^{At} = T \left[I + Dt + \frac{D^2t^2}{2!} + \dots + \frac{D^nt^n}{n!} \right] T^{-1}$$

Easy to compute!



Let's get back to x -coordinates

$$\dot{x} = Ax = T\dot{z}$$

$$x(t) = e^{At}x(0)$$

$$AD = TD$$

$$x = Tz$$

$$e^{At} = Te^{Dt}T^{-1}$$

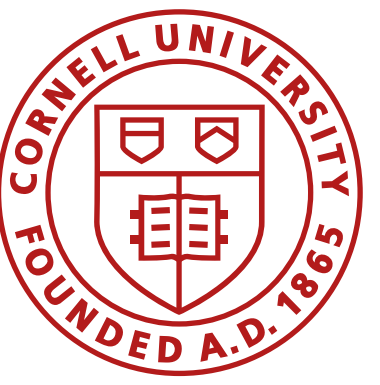
System solution in physical coordinates

$$x(t) = Te^{Dt}T^{-1}x(0)$$

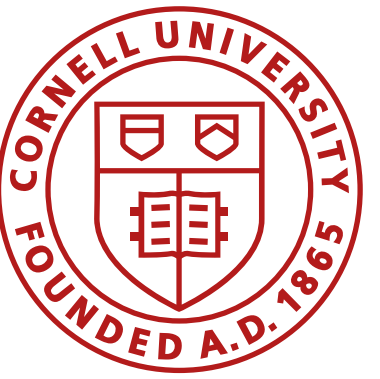
$$z(0)$$

$$z(t)$$

$$x(t)$$



Eigenvalues and Stability



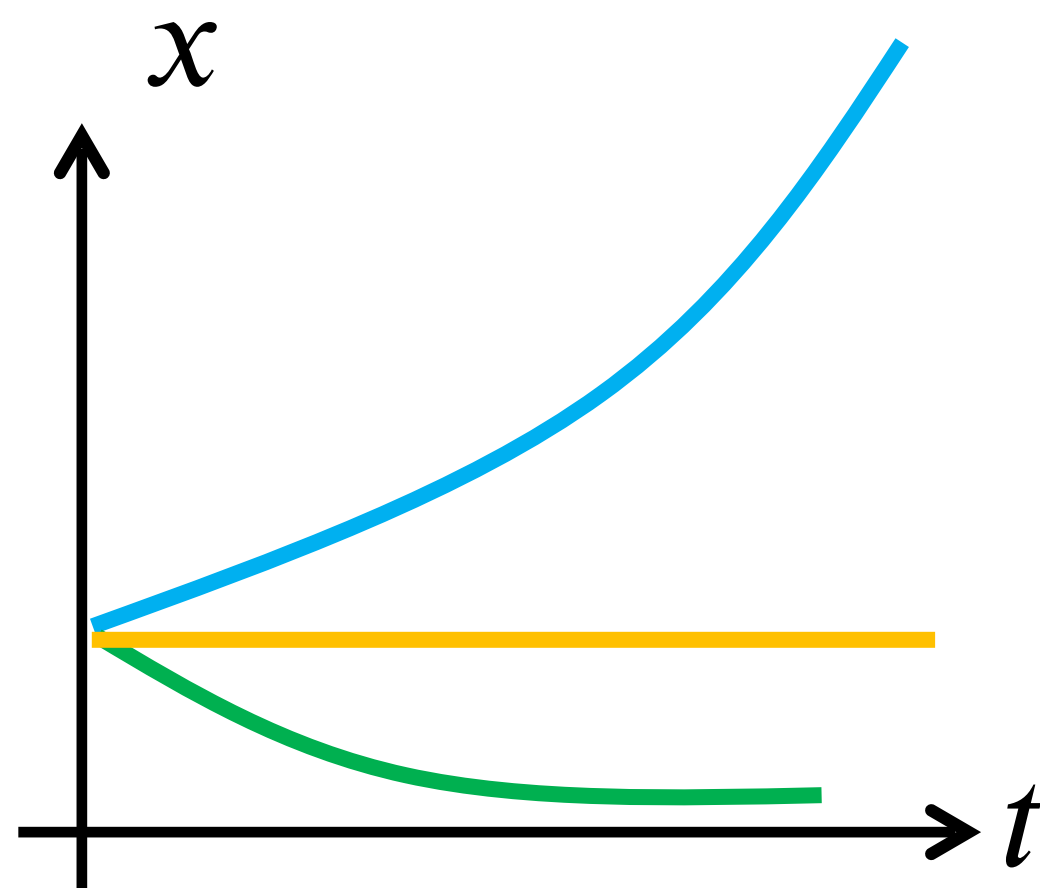
Stability (continuous time)

$$\dot{x} = Ax$$

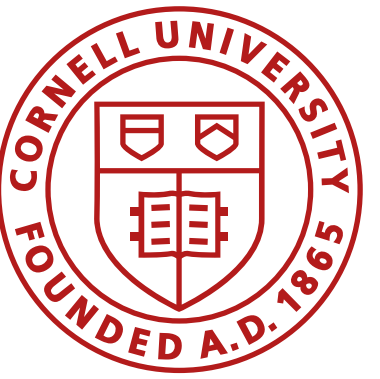
$$x(t) = Te^{Dt}T^{-1}x(0)$$

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix}$$

Python: eigenvalues, eigenvectors = np.linalg.eig(A)



- If even one of the $e^{\lambda t}$ goes to ∞ , all go to ∞
- Complex eigenvalues: $\lambda = a + ib$
- Euler's formula: $e^{\lambda t} = e^{at}(\cos(bt) + i \sin(bt))$



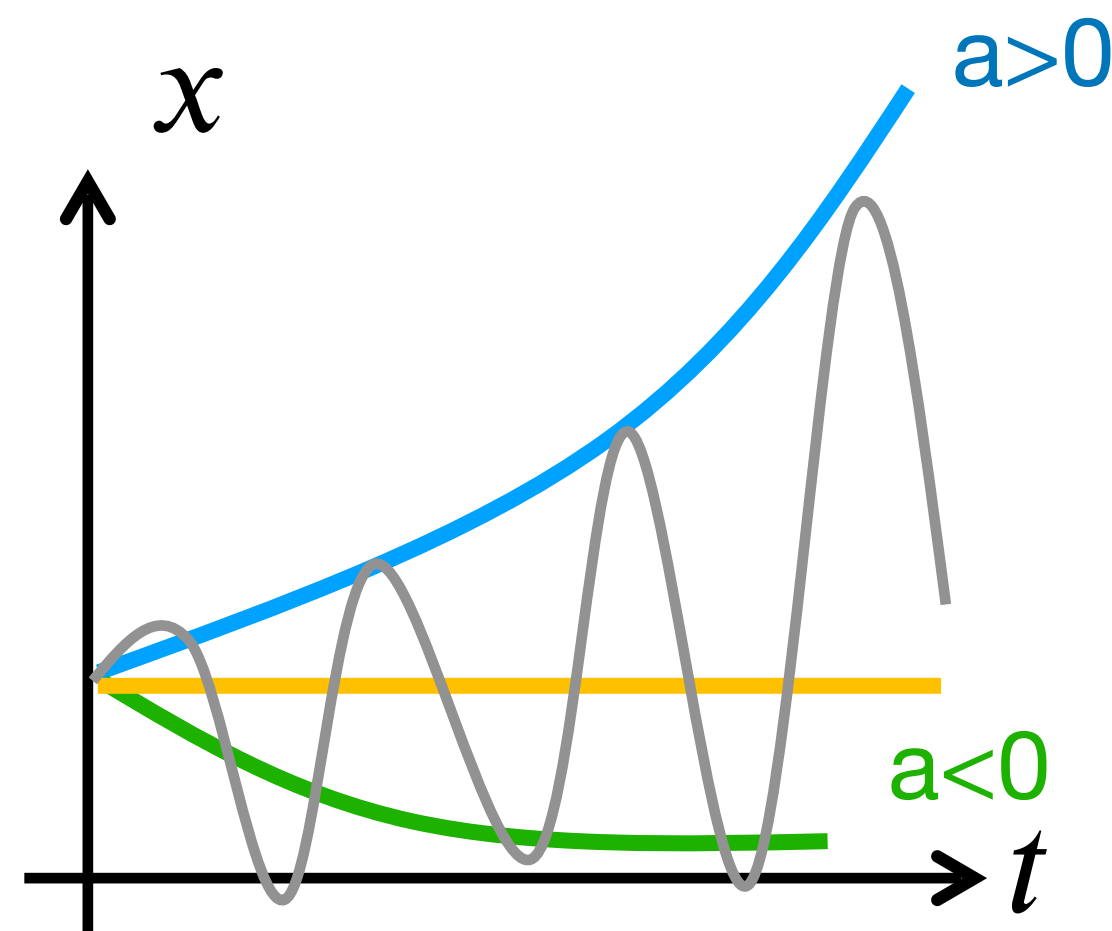
Stability (continuous time)

$$\dot{x} = Ax$$

$$x(t) = Te^{Dt}T^{-1}x(0)$$

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \quad e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & & & \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ & & & e^{\lambda_n t} \end{bmatrix}$$

Python: eigenvalues, eigenvectors = np.linalg.eig(A)



- If even one of the $e^{\lambda t}$ goes to ∞ , all go to ∞
- Complex eigenvalues: $\lambda = a \pm ib$

$$= 1$$
- Euler's formula: $e^{\pm \lambda t} = e^{at} (\cos(bt) \pm i \sin(bt))$

System is stable iff real parts of all eigenvalues are <0 !

Stability (continuous time)

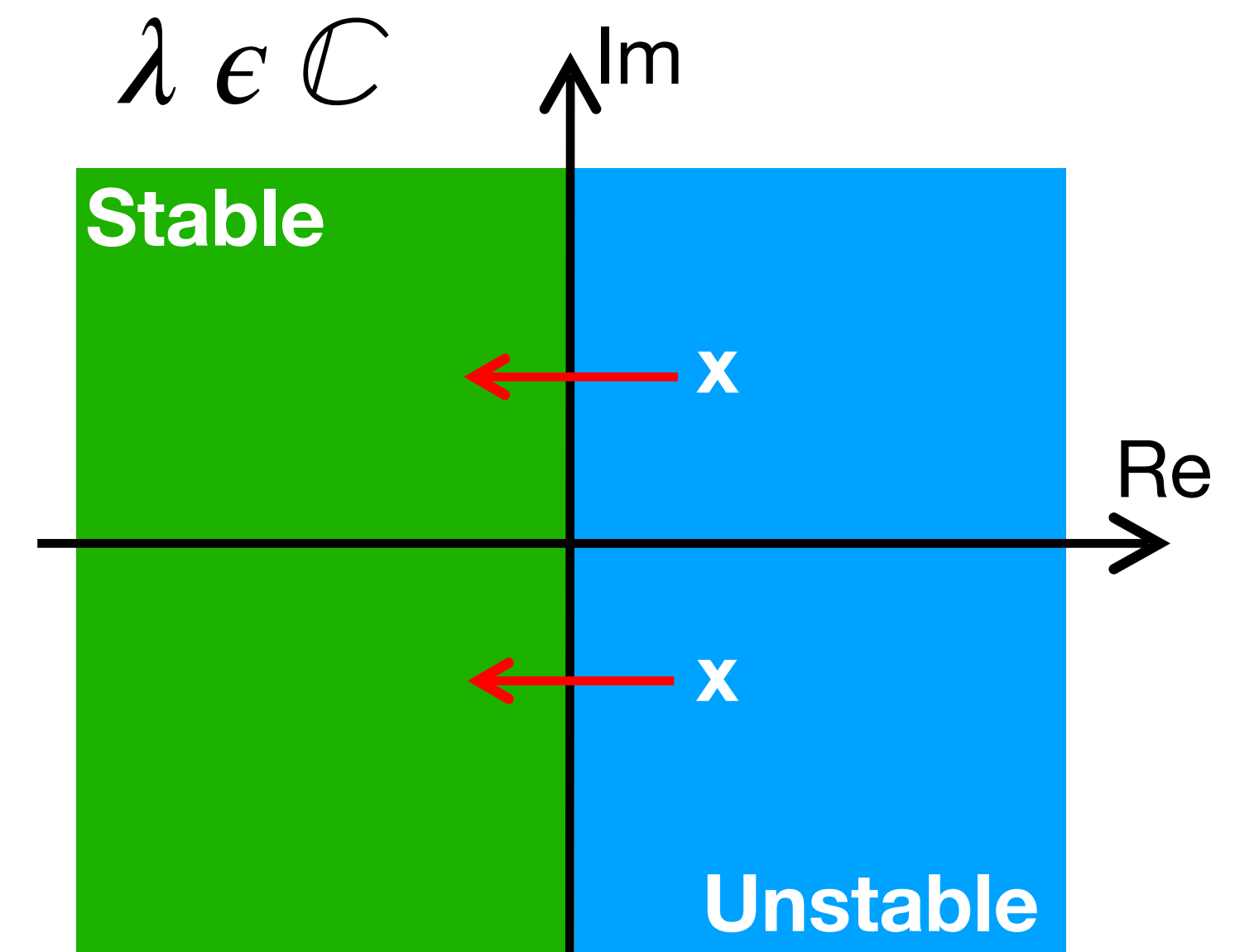
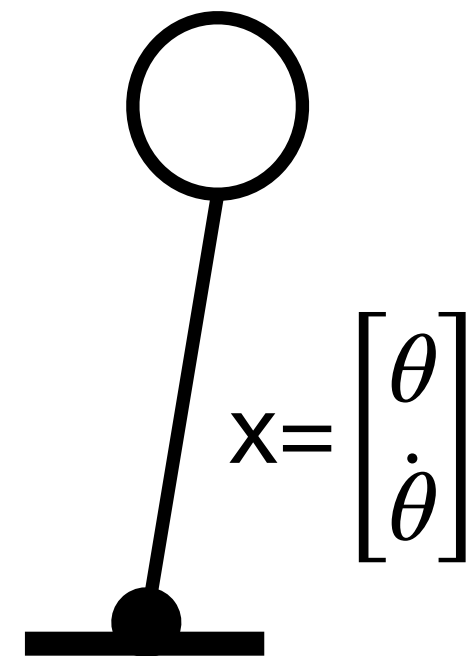
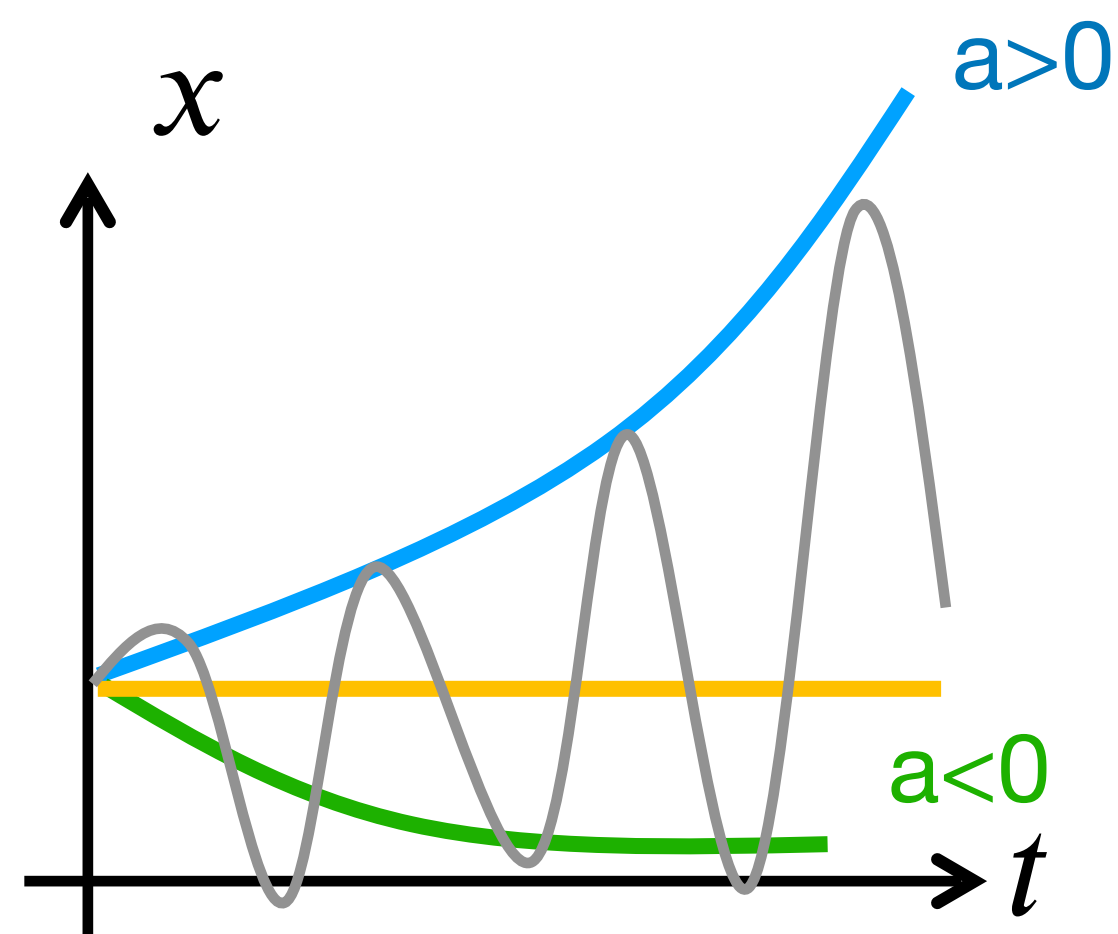
$$\dot{x} = Ax$$

$$x(t) = Te^{Dt}T^{-1}x(0)$$

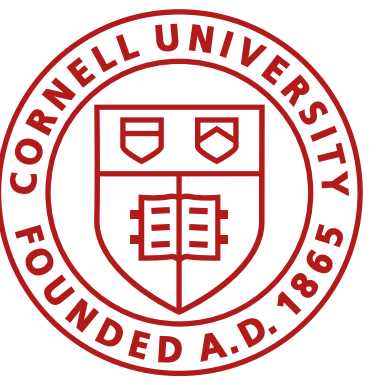
$$D = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{bmatrix}$$

$$\lambda = a + ib$$

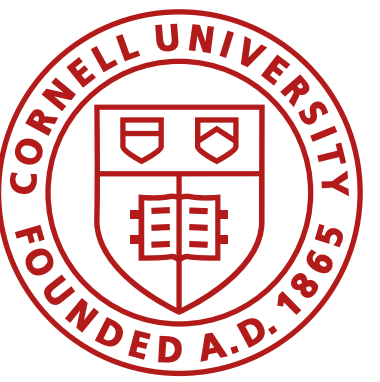
$$e^{Dt} = \begin{bmatrix} e^{\lambda_1 t} & & \\ & e^{\lambda_2 t} & \\ & & \ddots \\ & & & e^{\lambda_n t} \end{bmatrix}$$



System is stable iff real parts of all eigenvalues are < 0 !



Discrete Time Systems



Stability (discrete time)

$$\dot{x} = Ax$$

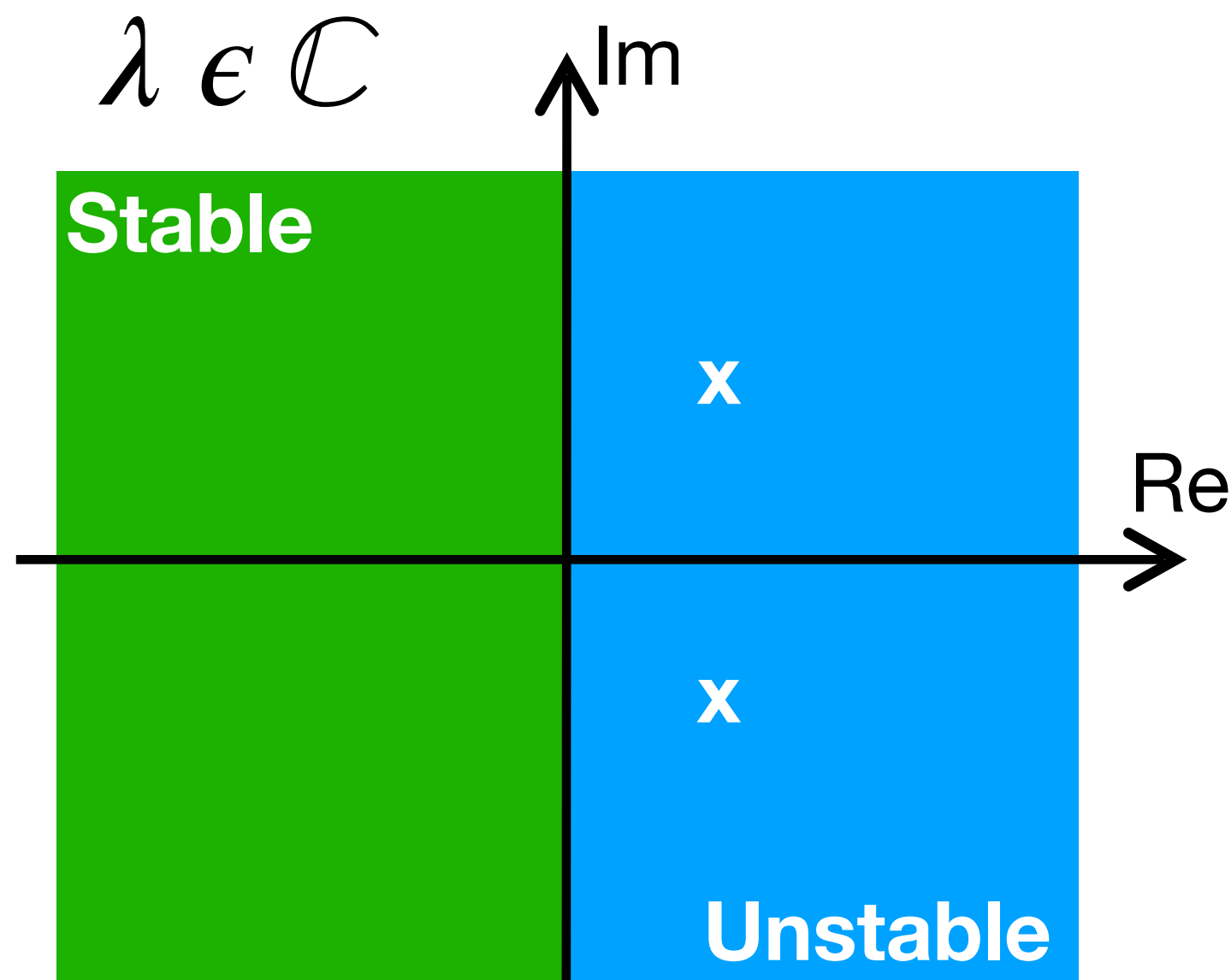
$$x_{k+1} = \tilde{A}x_k, \text{ where } x_k = x(k\Delta t)$$

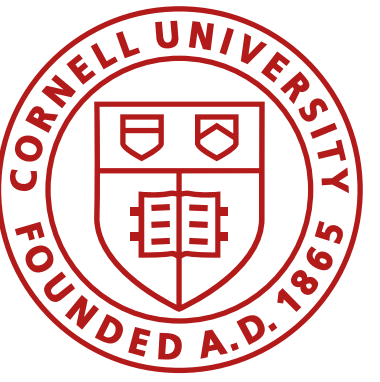
How does \tilde{A} relate to A ? $\tilde{A} = e^{A\Delta t}$

$x_1 = \tilde{A}x_0$	$\tilde{A} = \tilde{T}\tilde{D}\tilde{T}^{-1}$	$\tilde{\lambda}$
$x_2 = \tilde{A}x_1 = \tilde{A}^2x_0$	$\tilde{A}^2 = \tilde{T}\tilde{D}^2\tilde{T}^{-1}$	$\tilde{\lambda}^2$
$x_3 = \tilde{A}^3x_0$	$\tilde{A}^3 = \tilde{T}\tilde{D}^3\tilde{T}^{-1}$	$\tilde{\lambda}^3$
\vdots	\vdots	\vdots
$x_n = \tilde{A}^nx_0$	$\tilde{A}^n = \tilde{T}\tilde{D}^n\tilde{T}^{-1}$	$\tilde{\lambda}^n$

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$$\lambda \in \mathbb{C}$$

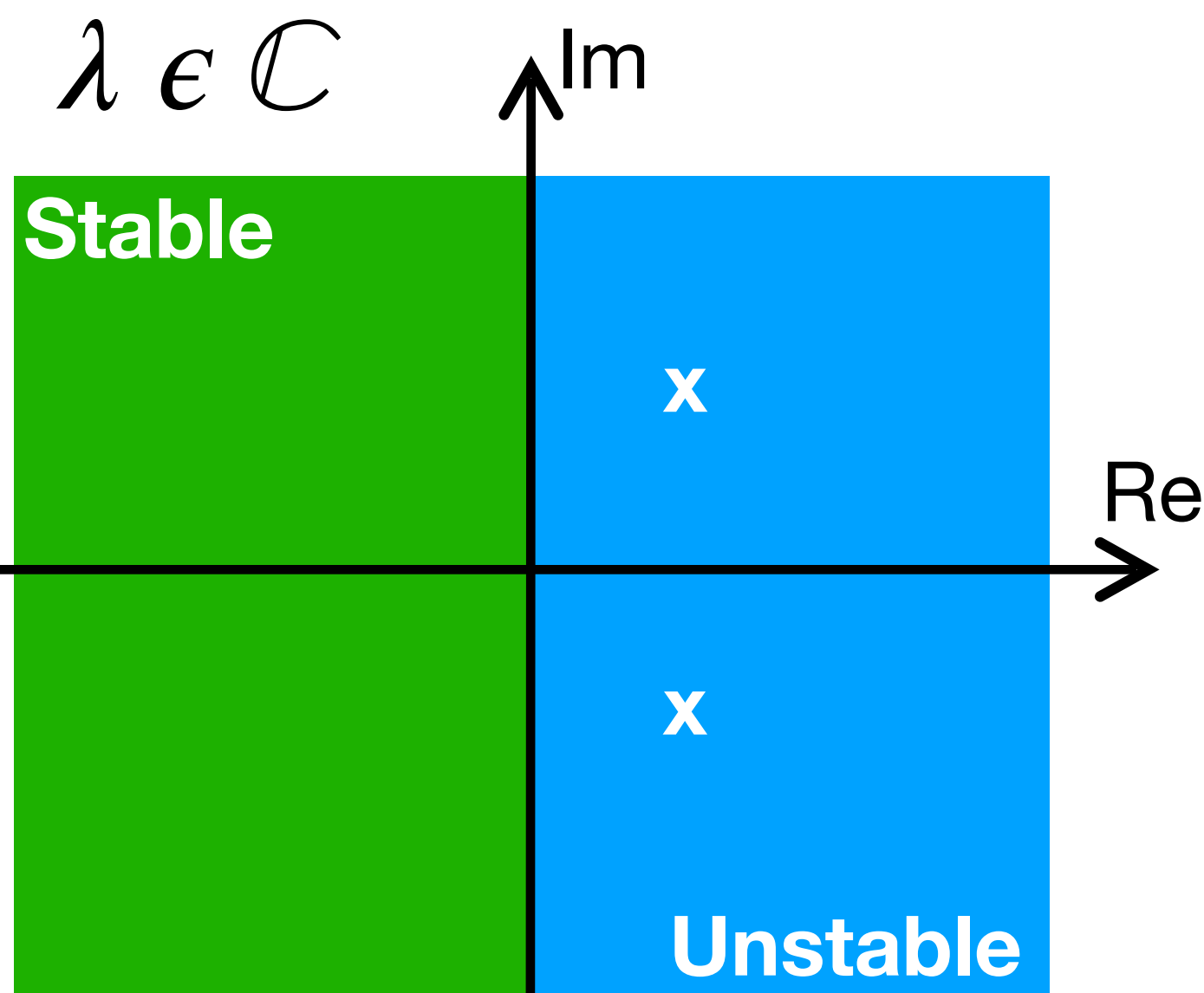




Stability (discrete time)

$$\dot{x} = Ax$$

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$



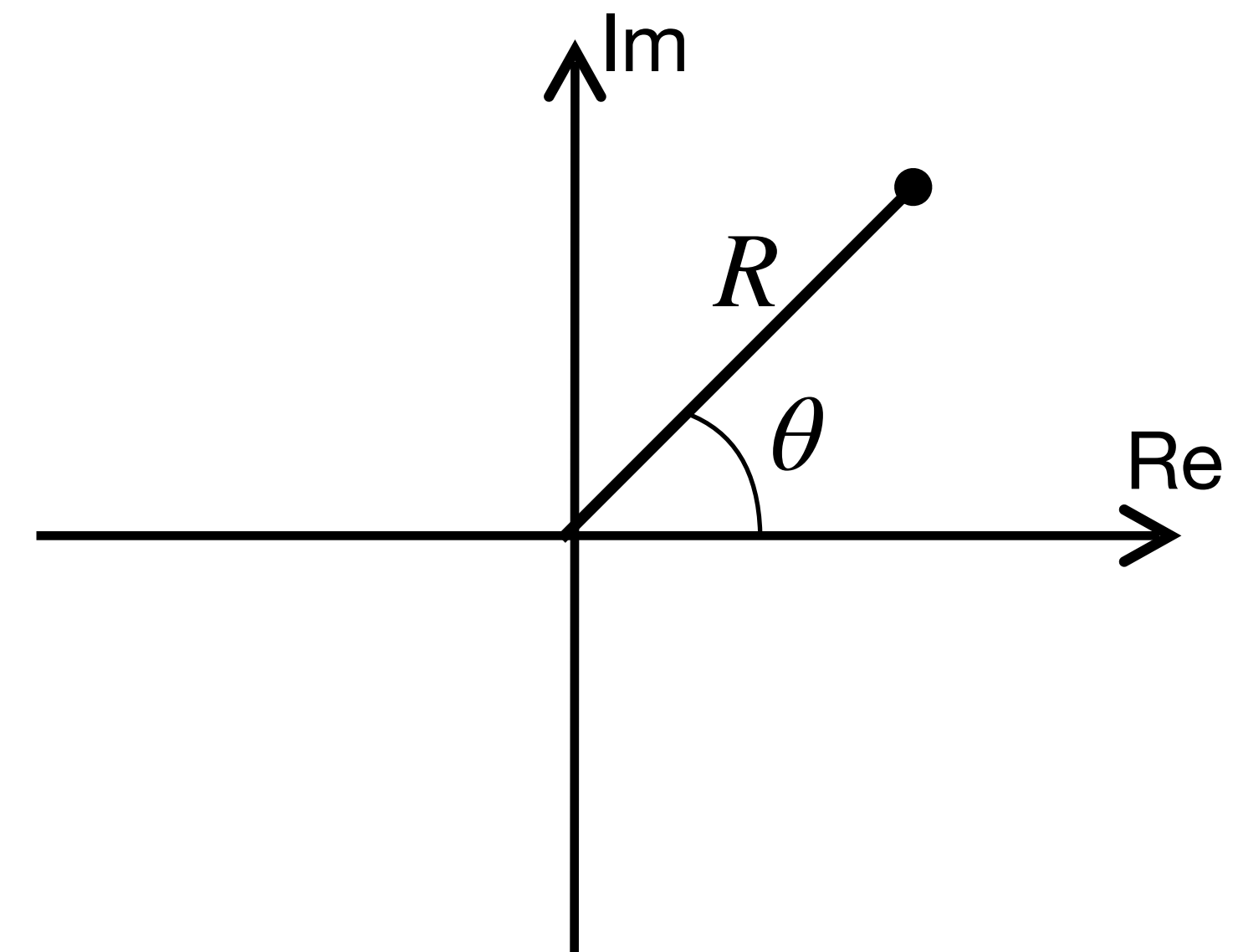
$$x_{k+1} = \tilde{A}x_k, \text{ where } x_k = x(k\Delta t)$$

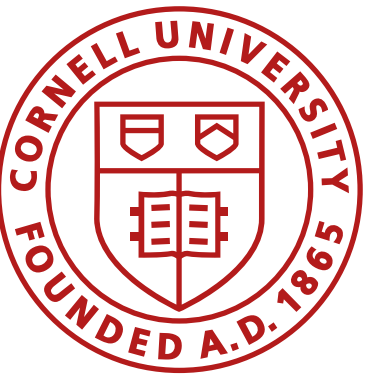
$$\text{How does } \tilde{A} \text{ relate to } A? \quad \tilde{A} = e^{A\Delta t}$$

$$x_n = \tilde{A}^n x_0 \quad \tilde{A}^n = \tilde{T} \tilde{D}^n \tilde{T}^{-1} \quad \tilde{\lambda}^n$$

$$\tilde{\lambda} = R e^{i\theta}$$

$$\tilde{\lambda}^n = R^n e^{in\theta}$$





Stability (discrete time)

$$\dot{x} = Ax$$

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

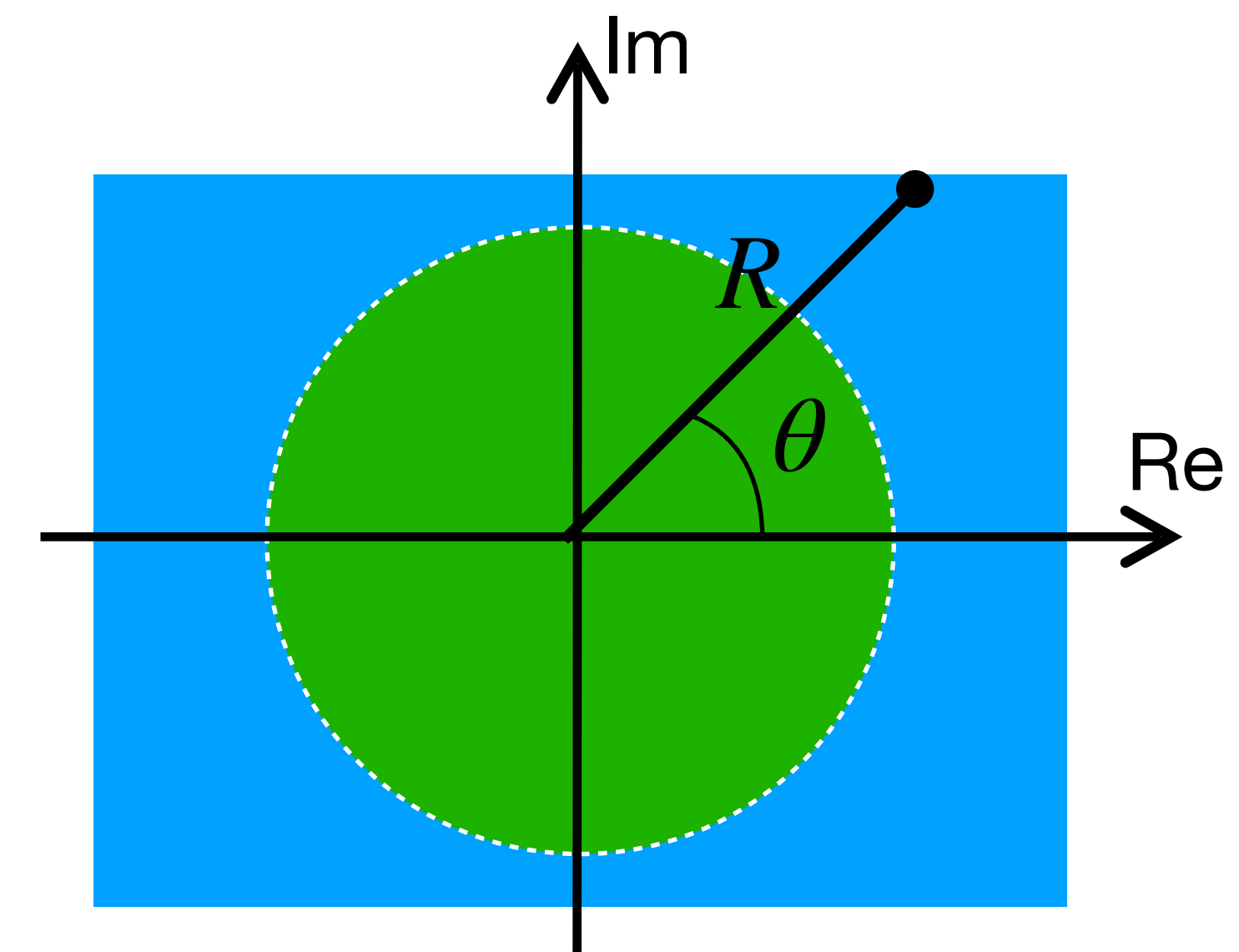
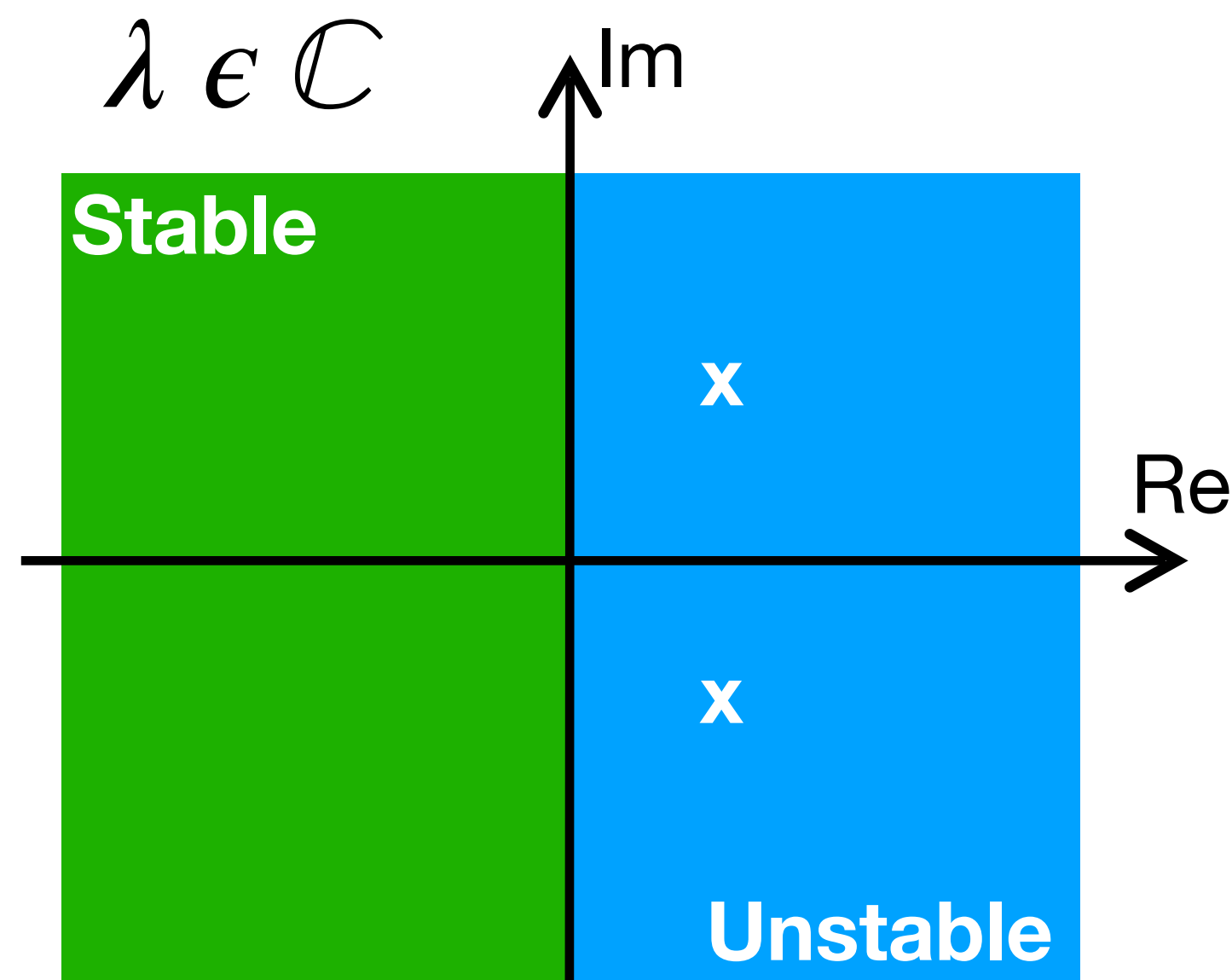
$$x_{k+1} = \tilde{A}x_k, \text{ where } x_k = x(k\Delta t)$$

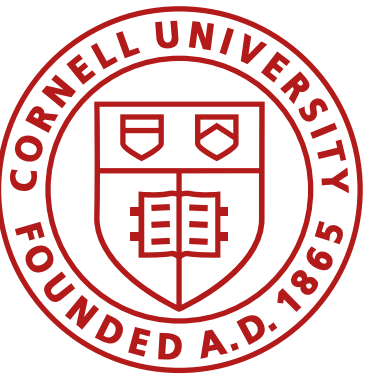
$$\text{How does } \tilde{A} \text{ relate to } A? \quad \tilde{A} = e^{A\Delta t}$$

$$x_n = \tilde{A}^n x_0 \quad \tilde{A}^n = \tilde{T} \tilde{D}^n \tilde{T}^{-1} \quad \tilde{\lambda}^n$$

$$\tilde{\lambda} = Re^{i\theta}$$

$$\tilde{\lambda}^n = R^n e^{in\theta}$$





Stability (discrete time)

$$\dot{x} = Ax$$

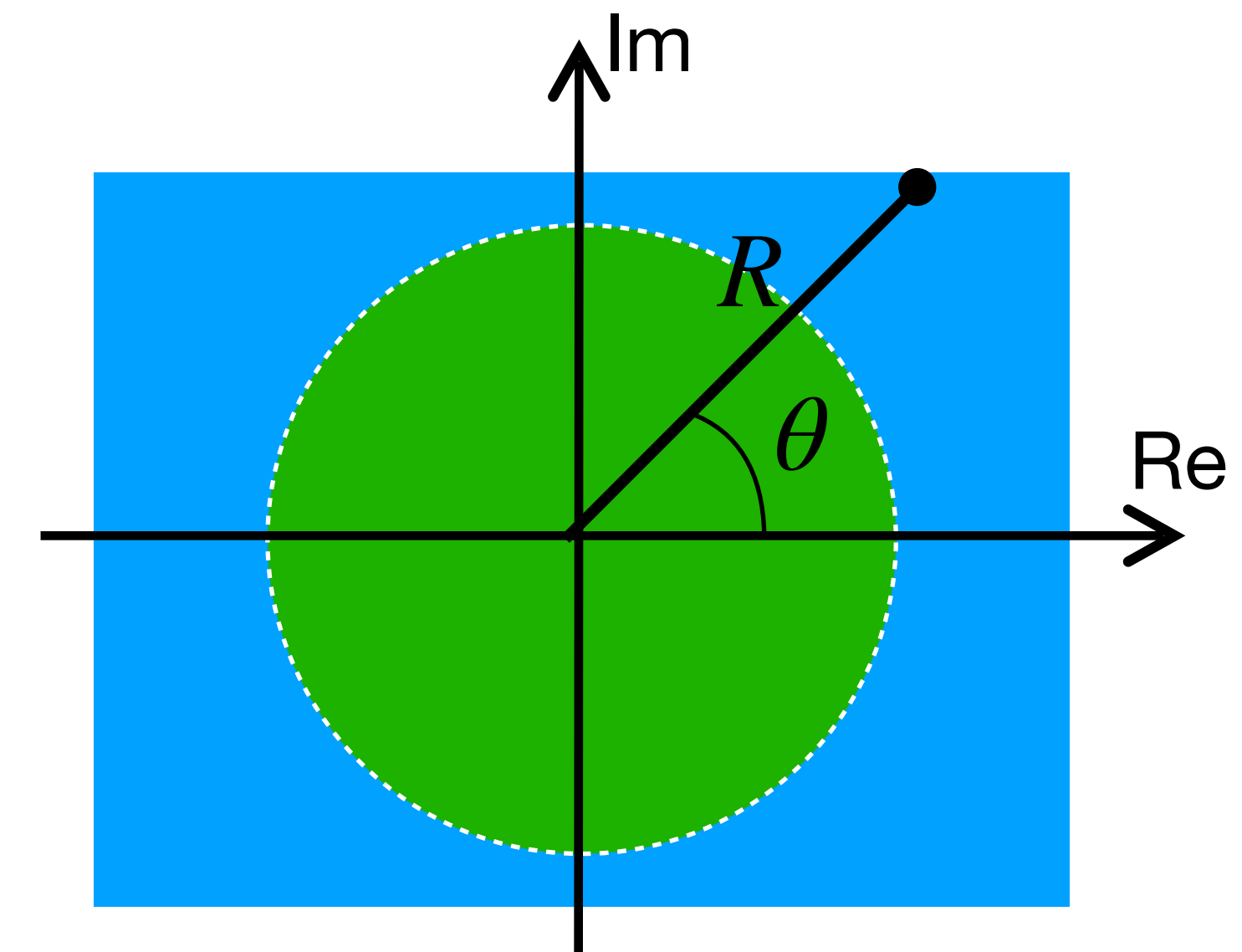
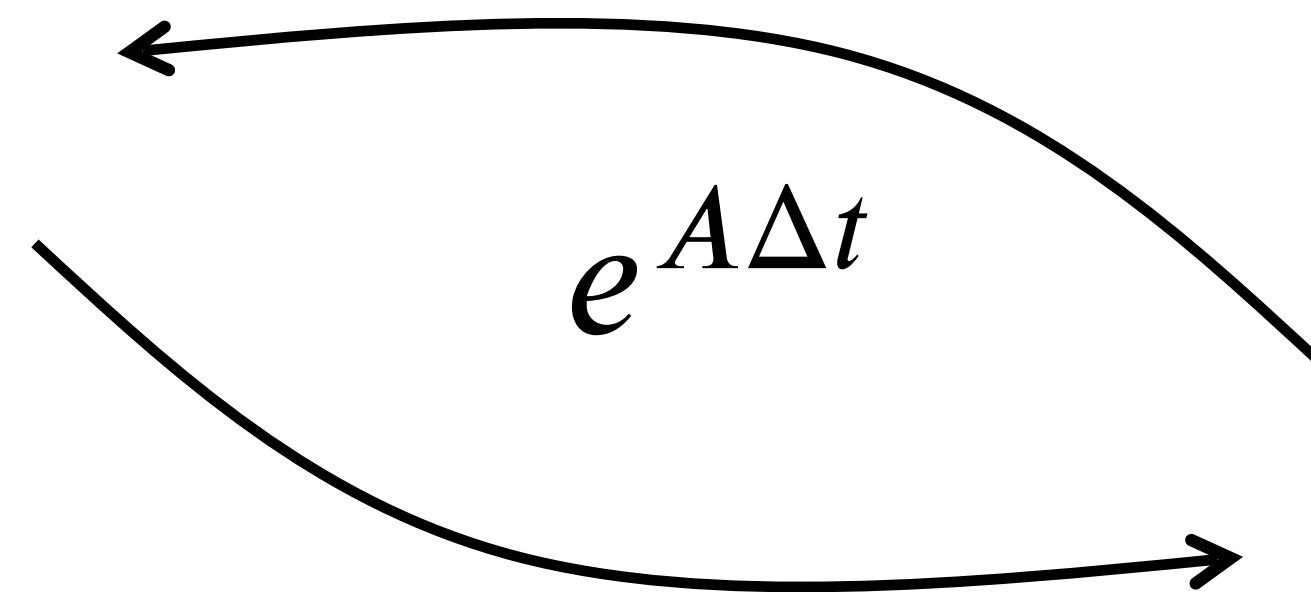
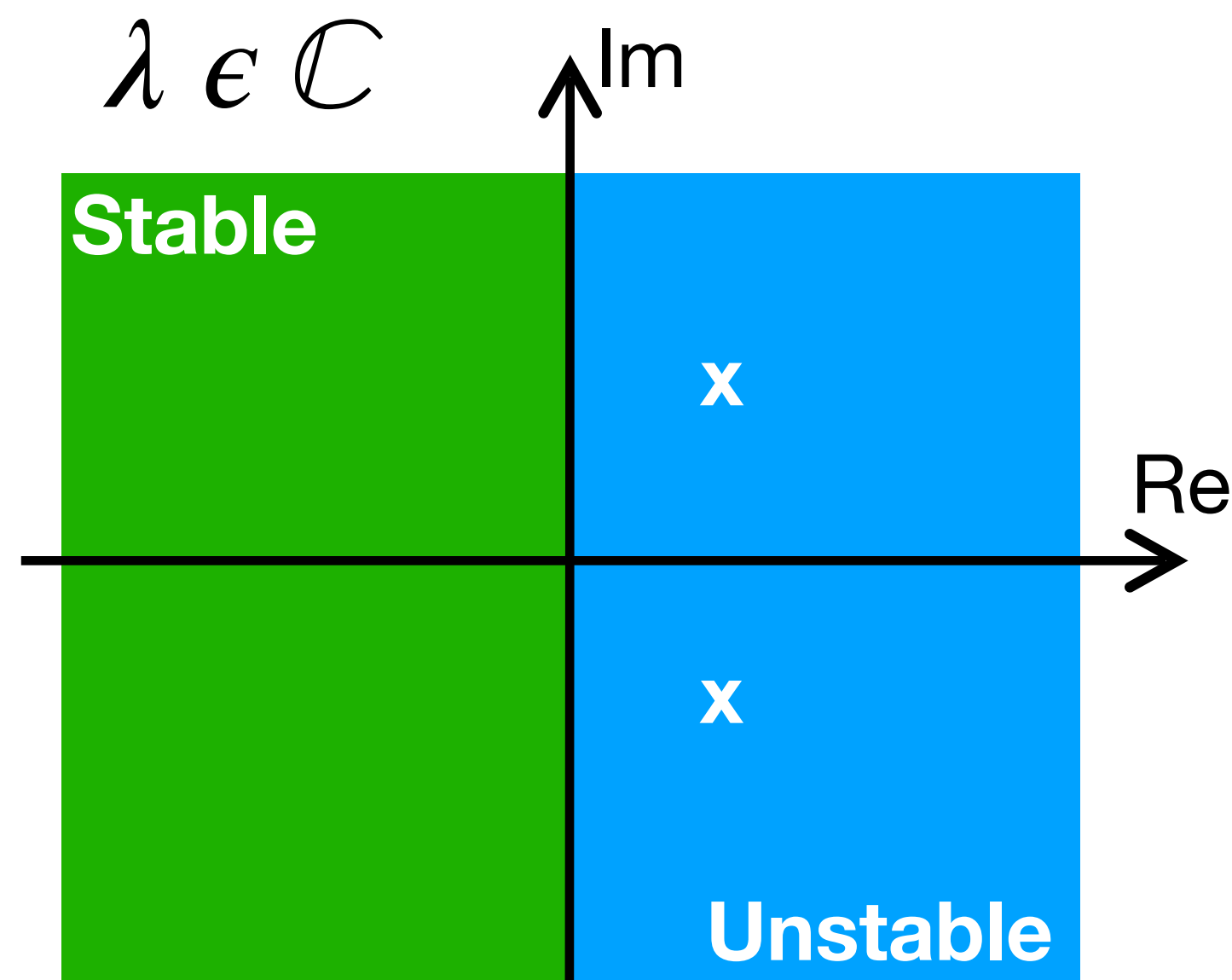
$$x_{k+1} = \tilde{A}x_k, \text{ where } x_k = x(k\Delta t)$$

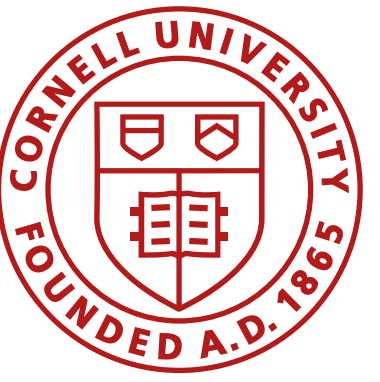
$$\text{How does } \tilde{A} \text{ relate to } A? \quad \tilde{A} = e^{A\Delta t}$$

$$x_n = \tilde{A}^n x_0 \quad \tilde{A}^n = \tilde{T} \tilde{D}^n \tilde{T}^{-1} \quad \tilde{\lambda}^n$$

$$\tilde{\lambda} = Re^{i\theta}$$

$$\tilde{\lambda}^n = R^n e^{in\theta}$$





Stability (discrete time)

$$\dot{x} = Ax$$

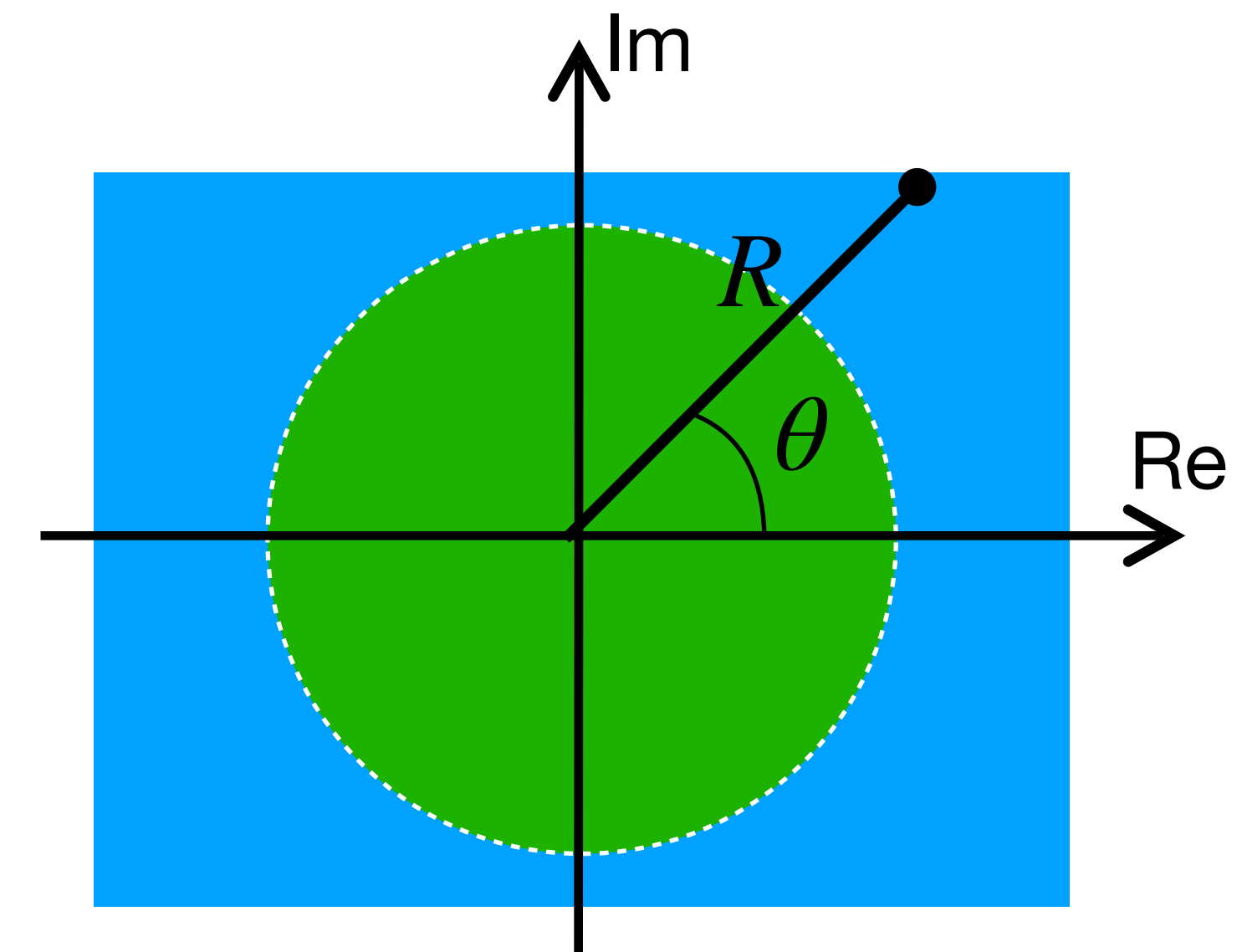
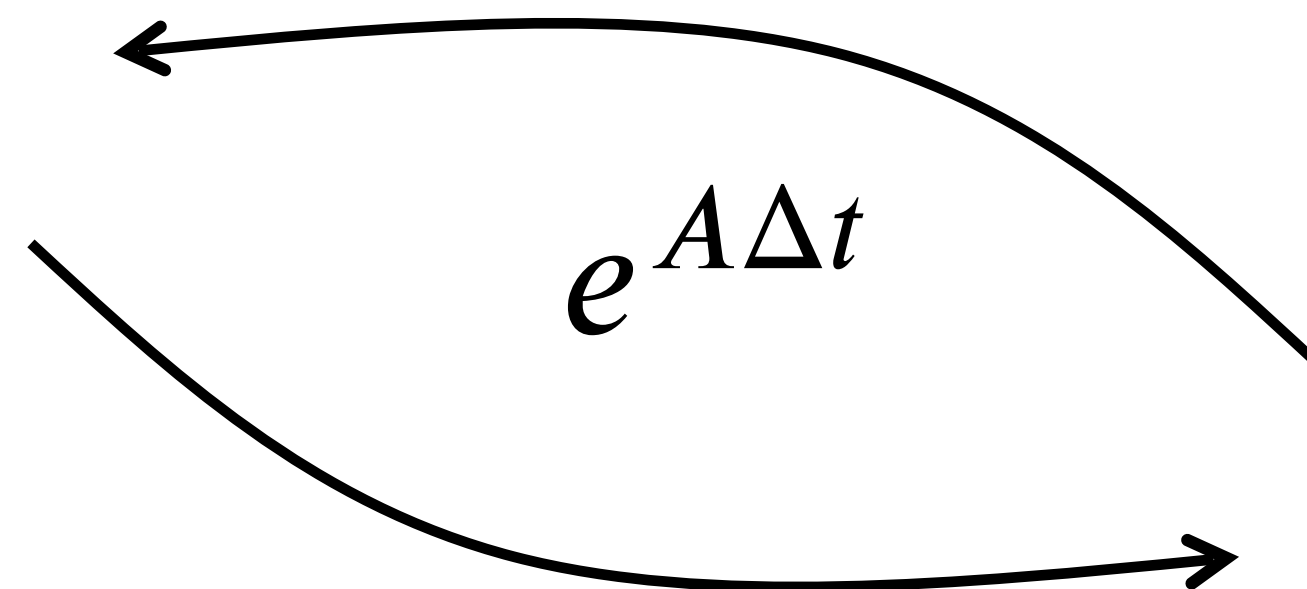
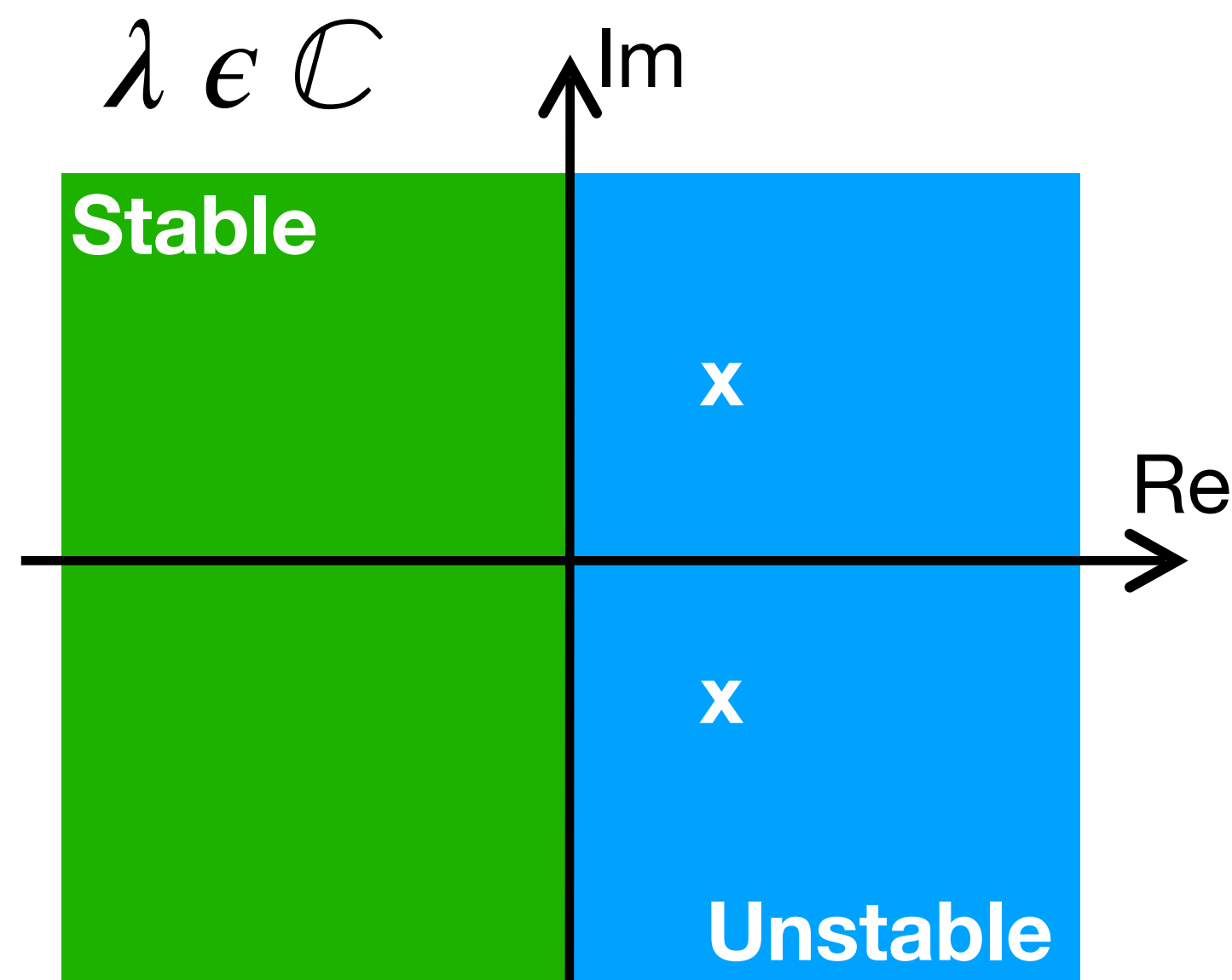
$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

$$x_{k+1} = \tilde{A}x_k$$

$$\tilde{A} = e^{A\Delta t}$$

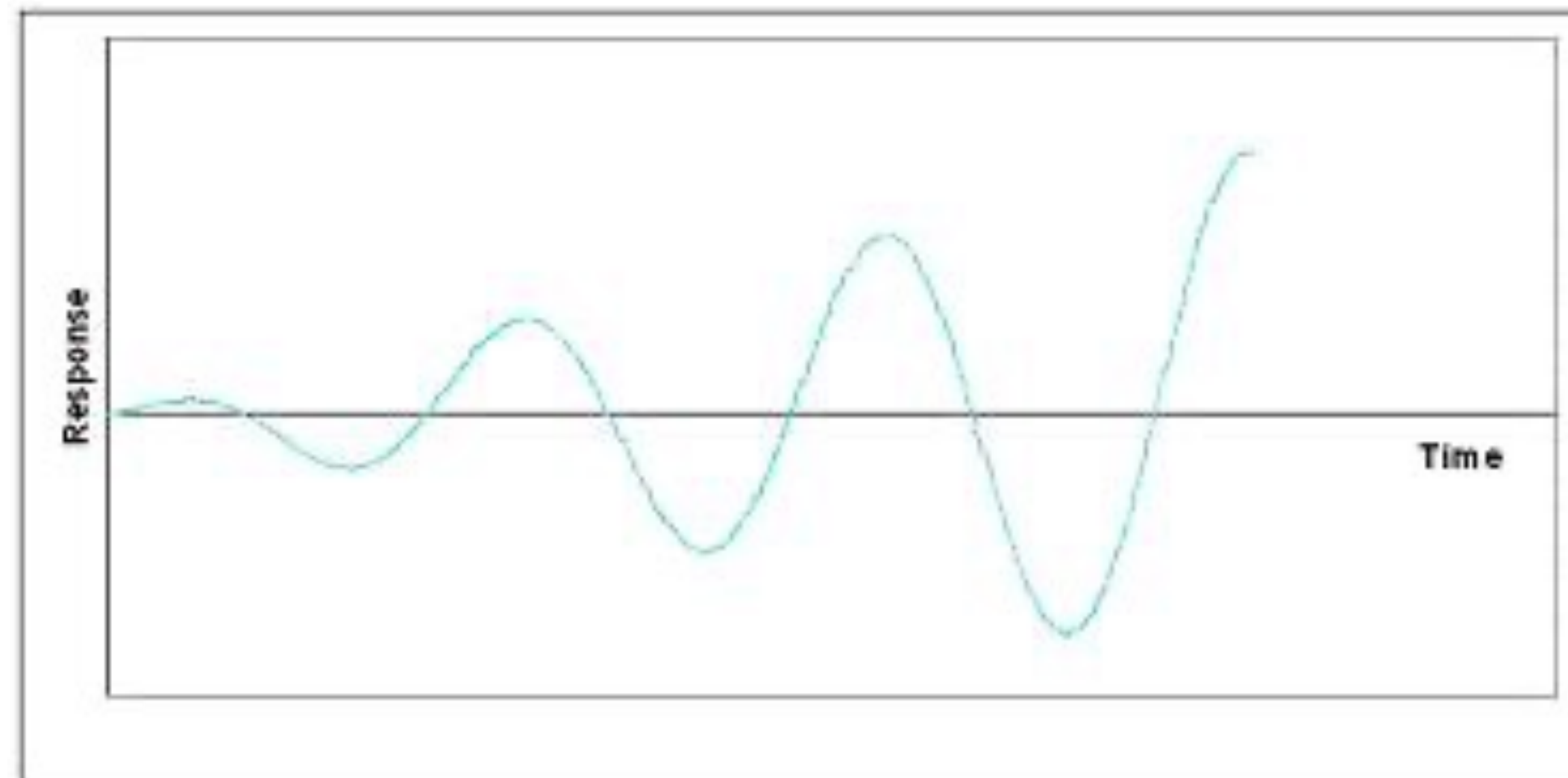
$$\tilde{\lambda}^n = R^n e^{in\theta}$$

- We often work in discrete time
- Stability and quality of controllers depend on sampling rate



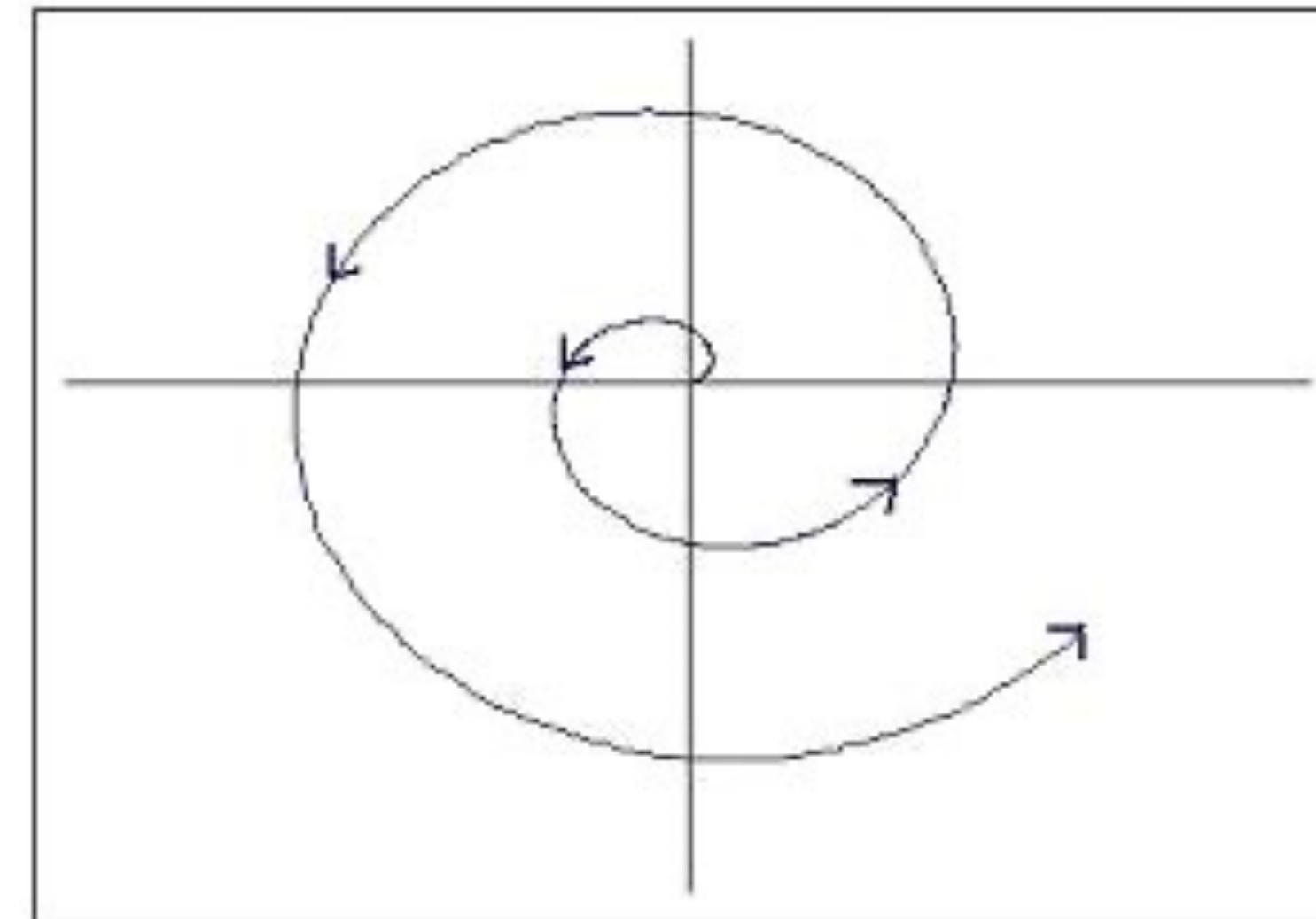
Stability (discrete time)

$$\dot{x} = Ax$$



Unstable
(Positive real part, $R > 1$)

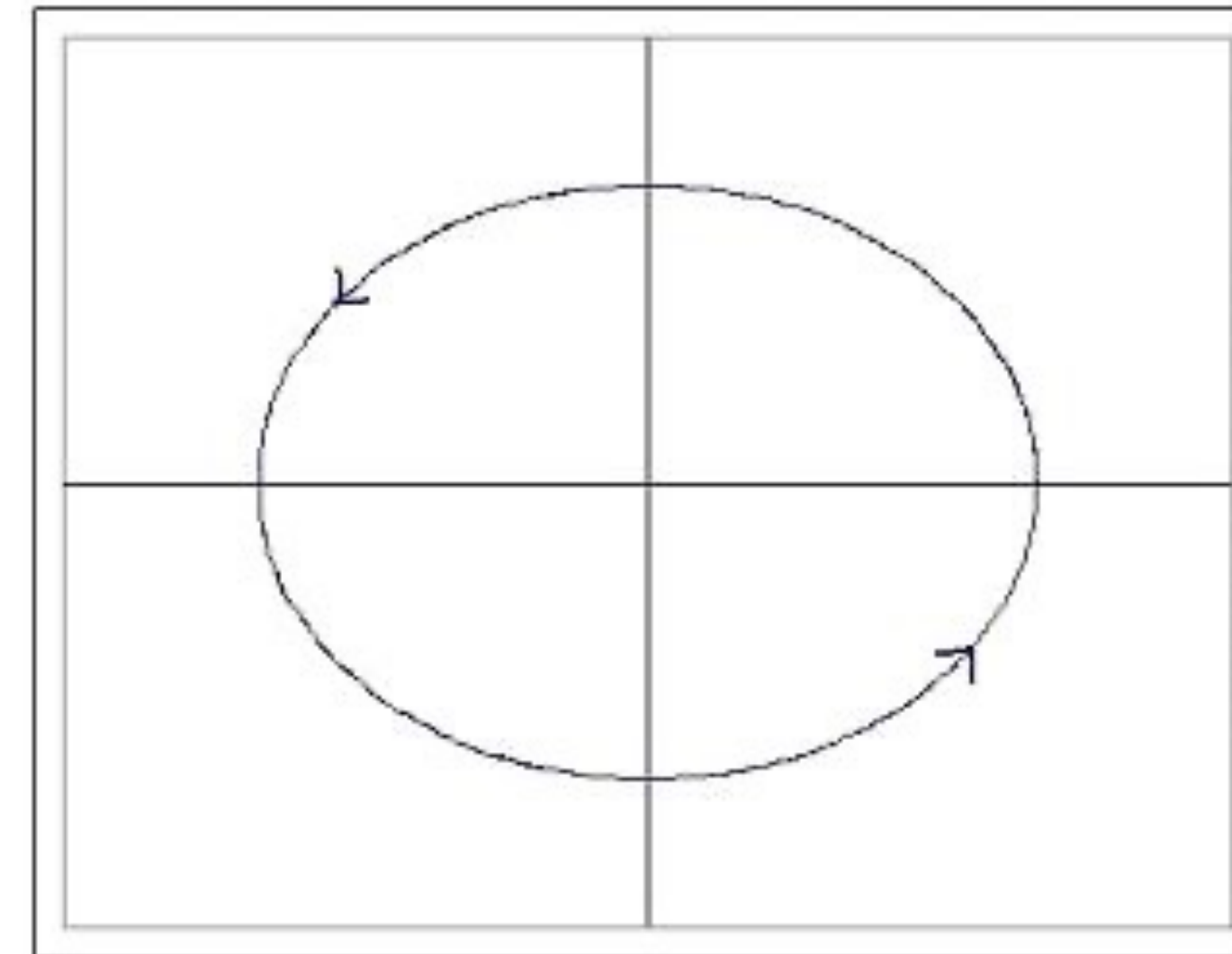
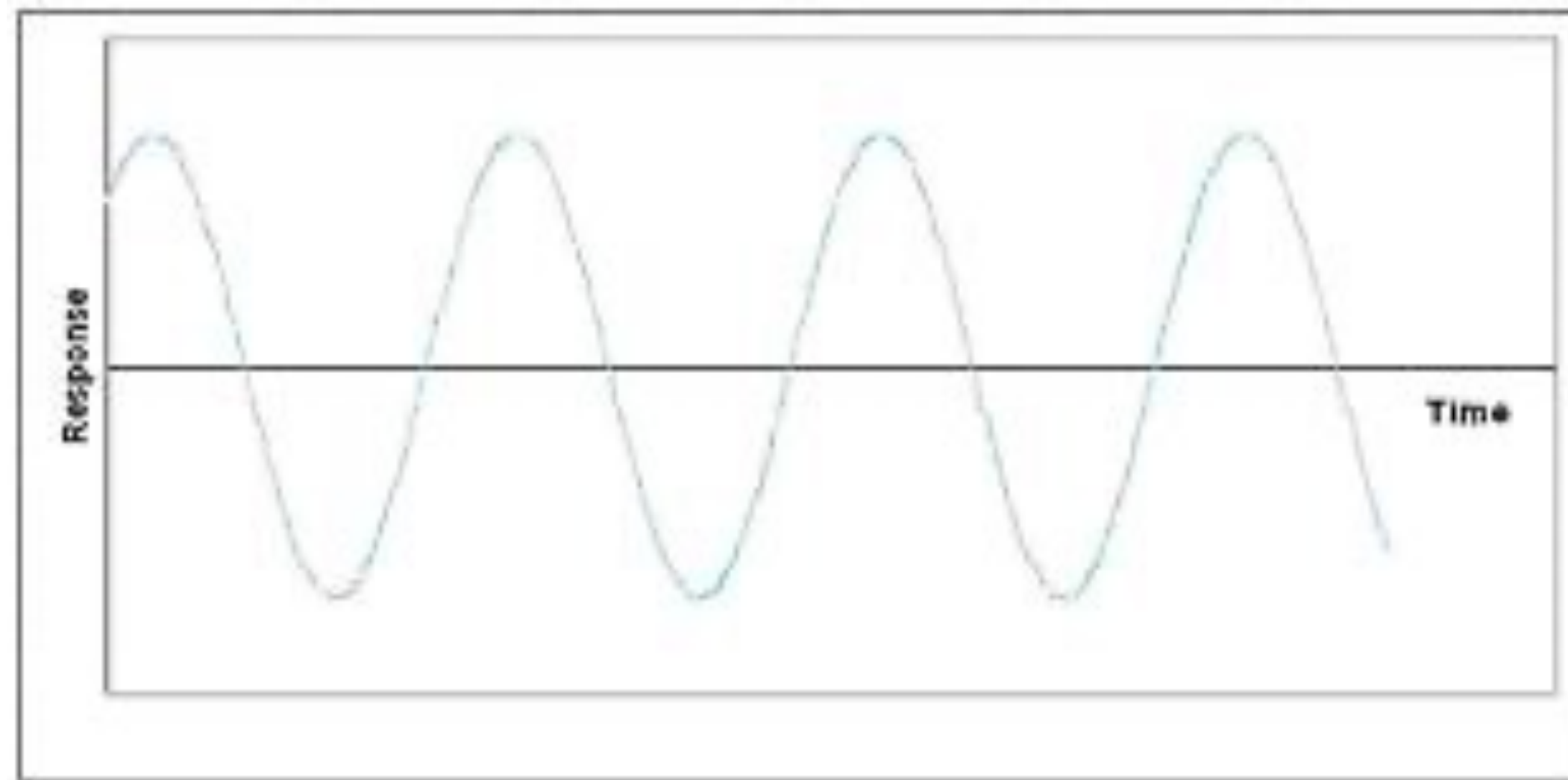
$$x_{k+1} = \tilde{A}x_k$$



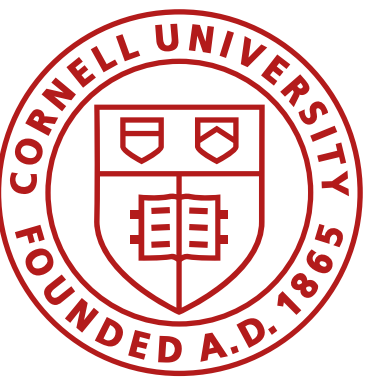
Stability (discrete time)

$$\dot{x} = Ax$$

$$x_{k+1} = \tilde{A}x_k$$



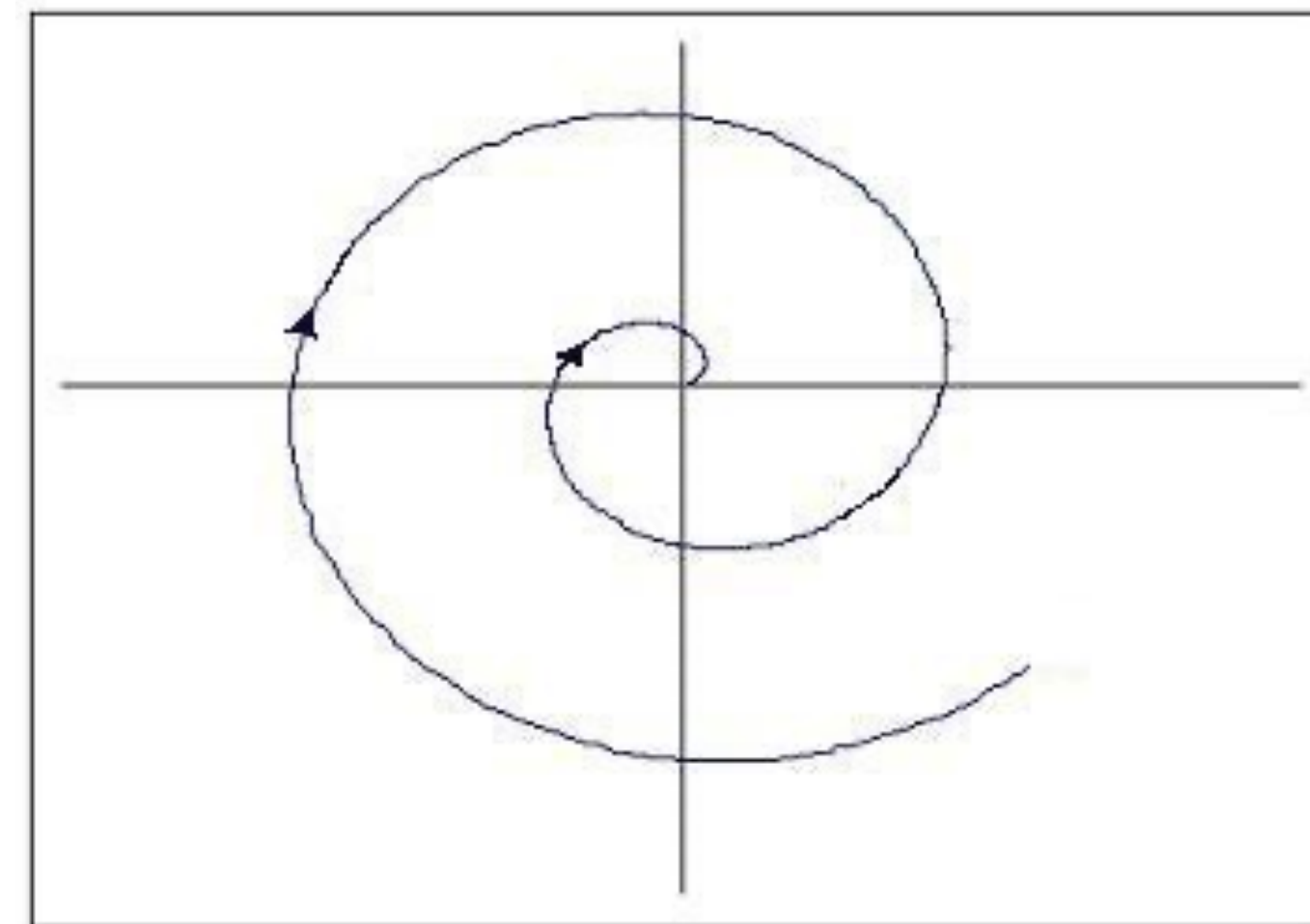
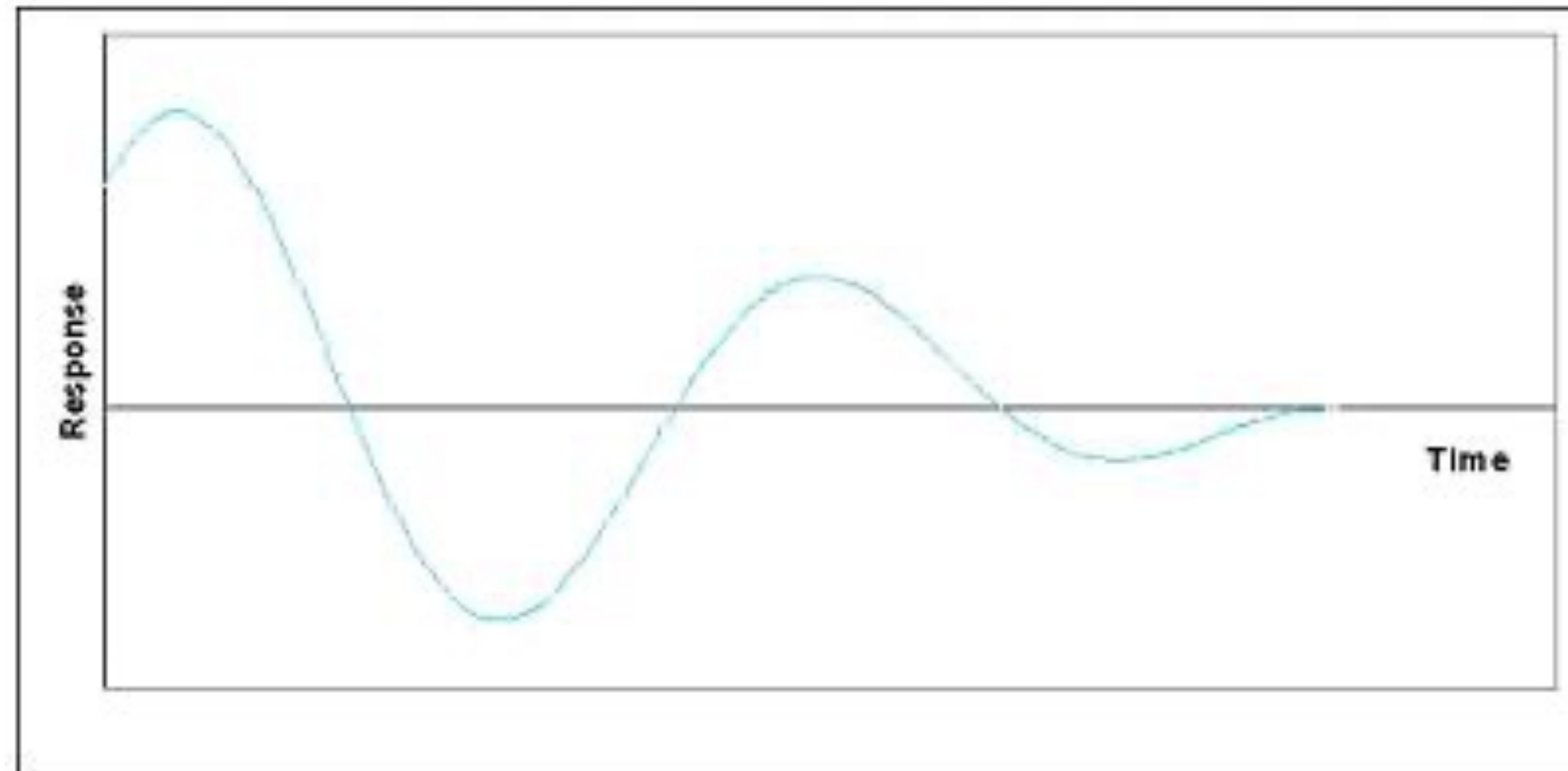
Critically stable
(Zero real part, $R = 1$)



Stability (discrete time)

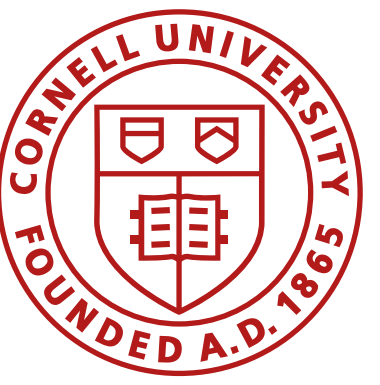
$$\dot{x} = Ax$$

$$x_{k+1} = \tilde{A}x_k$$



Stable

(Negative real part, $R < 1$)



Linearizing Nonlinear Systems

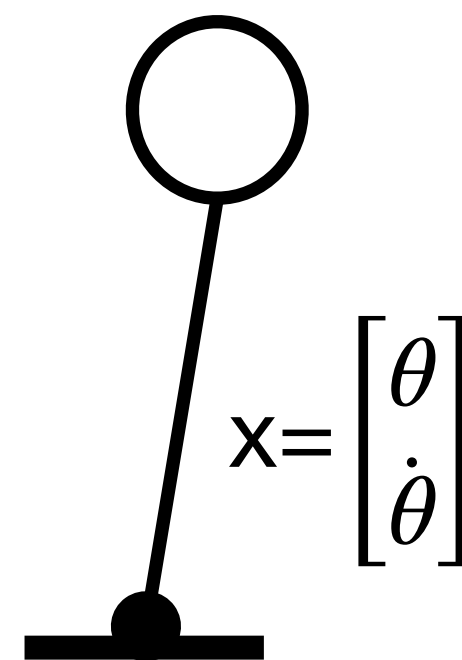
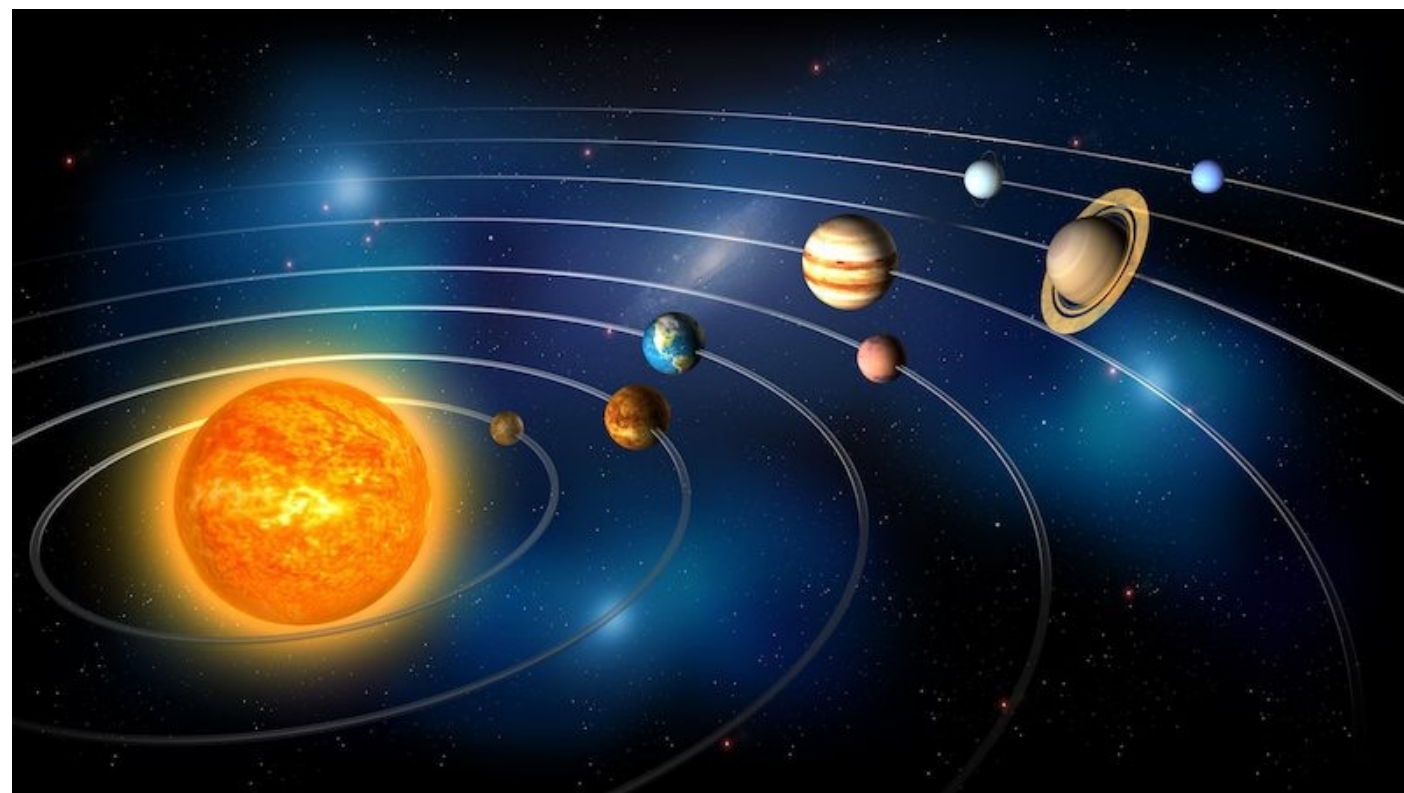
Basic steps to linearize nonlinear systems

- Find some fixed points

- \bar{x} st $f(\bar{x}) = 0$

- Linearize about them

- $\left. \frac{Df}{Dx} \right|_{\bar{x}} = \begin{bmatrix} \frac{\delta f_i}{\delta x_j} \end{bmatrix}$ "Jacobian"



$$\dot{x} = f(x) \longrightarrow \dot{x} = Ax$$

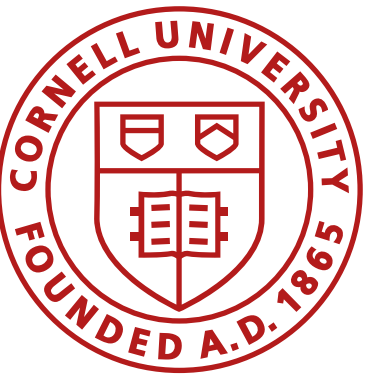
Example:

$$\dot{x}_1 = f_1(x_1, x_2) = x_1 x_2$$

$$\dot{x}_2 = f_2(x_1, x_2) = x_1^2 + x_2^2$$

$$\frac{Df}{Dx} = \begin{bmatrix} \frac{\delta f_1}{\delta x_1} & \frac{\delta f_1}{\delta x_2} \\ \frac{\delta f_2}{\delta x_1} & \frac{\delta f_2}{\delta x_2} \end{bmatrix}$$

$$\frac{Df}{Dx} = \begin{bmatrix} x_2 & x_1 \\ 2x_1 & 2x_2 \end{bmatrix} \text{ Evaluate at } \bar{x}$$

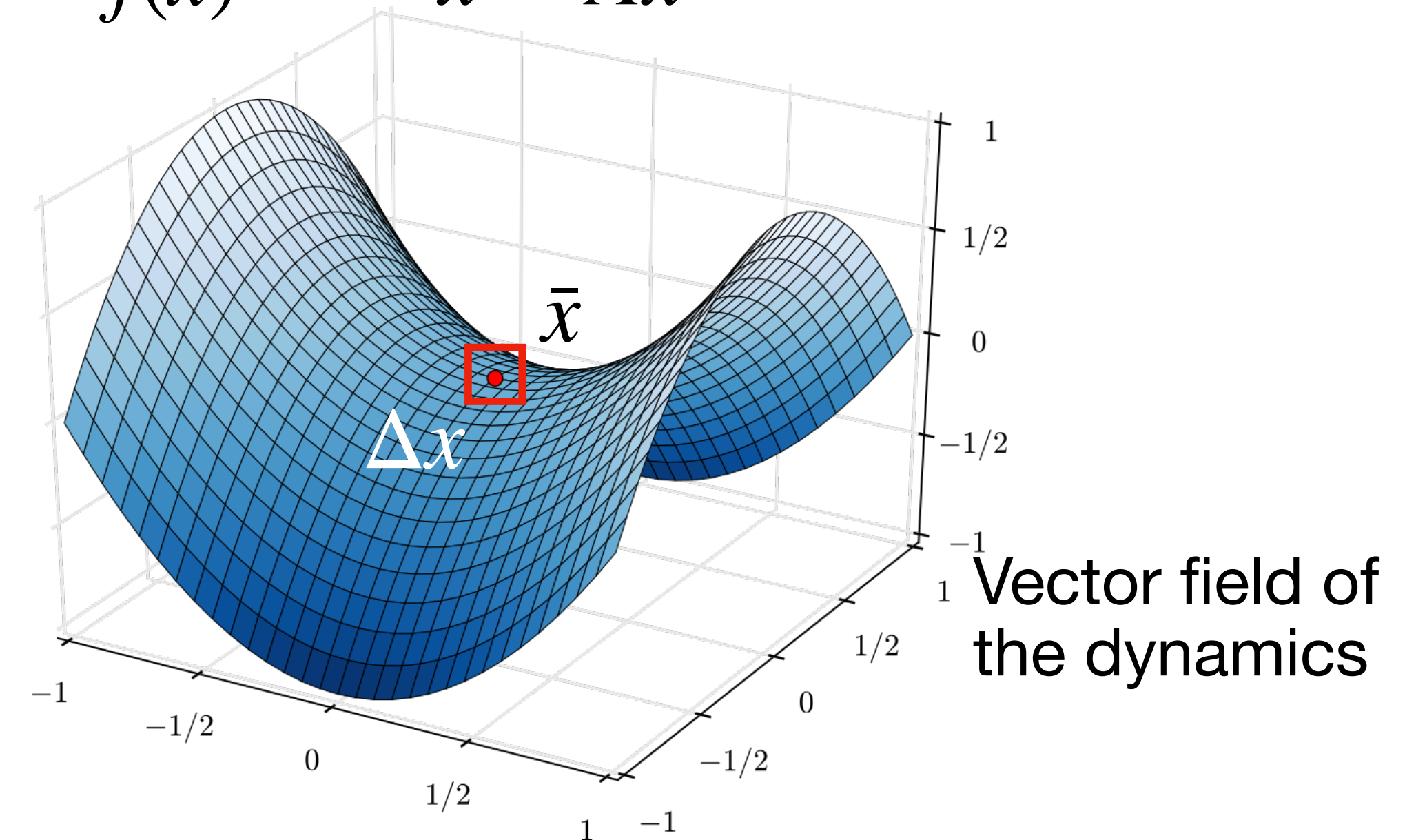


Basic steps to linearize nonlinear systems

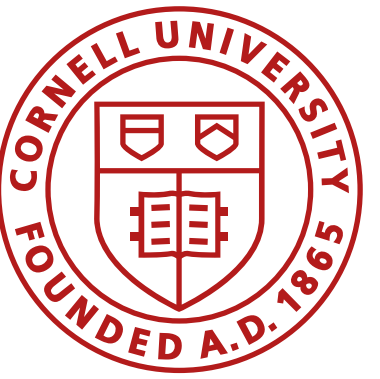
$$\dot{x} = f(x) \longrightarrow \dot{x} = Ax$$

- Find some fixed points
 - \bar{x} st $f(\bar{x}) = 0$
- Linearize about them

$$\frac{Df}{Dx} \Big|_{\bar{x}} = \begin{bmatrix} \frac{\delta f_i}{\delta x_j} \end{bmatrix}$$



$$\dot{x} = f(\bar{x}) + \frac{Df}{Dx} \Big|_{\bar{x}} (x - \bar{x}) + \frac{D^2f}{D^2x} \Big|_{\bar{x}} (x - \bar{x})^2 + \frac{D^3f}{D^3x} \Big|_{\bar{x}} (x - \bar{x})^3 + \dots$$



Basic steps to linearize nonlinear systems

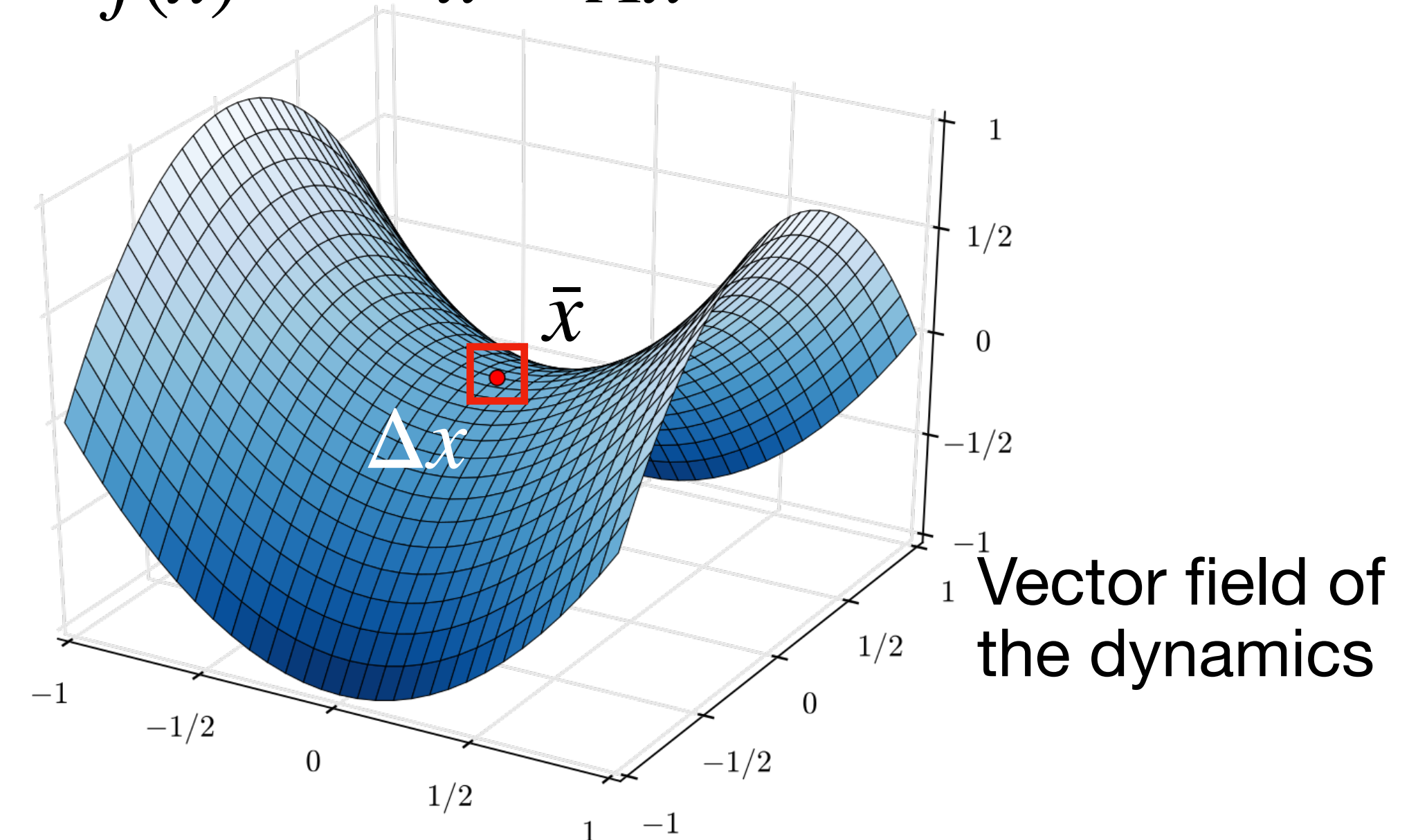
- Find some fixed points
 - \bar{x} st $f(\bar{x}) = 0$

- Linearize about them

$$\frac{Df}{Dx} \Big|_{\bar{x}} = \begin{bmatrix} \frac{\delta f_i}{\delta x_j} \end{bmatrix}$$

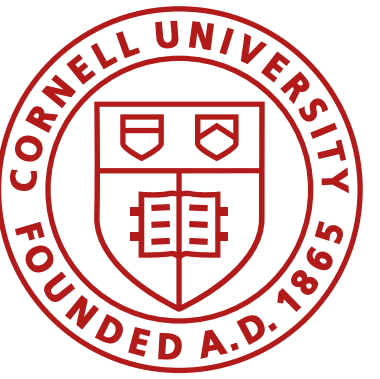
- If you zoom in on \bar{x} , your system will look linear!
- Good control will keep you near the fixed point, where the model is valid!

$$\dot{x} = f(x) \longrightarrow \dot{x} = Ax$$



$$\dot{x} = \overset{0}{f(\bar{x})} + \frac{Df}{Dx} \Big|_{\bar{x}} (x - \bar{x}) + \frac{D^2f}{D^2x} \Big|_{\bar{x}} (x - \bar{x})^2 + \frac{D^3f}{D^3x} \Big|_{\bar{x}} (x - \bar{x})^3 + \dots \lll 1$$

$$\Delta \dot{x} = \frac{Df}{Dx} \Big|_{\bar{x}} (\Delta x) \longrightarrow \boxed{\Delta \dot{x} = A \Delta x}$$



Review

- Linear system: $\dot{x} = Ax$
- Solution: $x(t) = e^{At}x(0)$
- Eigenvectors: $T = [\xi_1 \quad \xi_2 \quad \dots \quad \xi_n]$
- Eigenvalues: $D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$
- Linear Transform: $AT = TD$
- Solution: $e^{At} = e^{TDT^{-1}t}$
- Mapping from x to z to x : $x(t) = Te^{Dt}T^{-1}x(0)$
- Stability in continuous time: $\lambda = a + ib$, stable iff $a < 0$
- Discrete time: $x(k + 1) = \tilde{A}x(k)$, where $\tilde{A} = e^{A\Delta t}$
- Stability in discrete time: $\tilde{\lambda}^n = R^n e^{in\theta}$, stable iff $R < 1$
- Nonlinear systems: $\dot{x} = f(x)$
- Linearization: $\left. \frac{Df}{Dx} \right|_{\bar{x}}$