Prof. Kirstin Hagelskjær Petersen
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ECE 4160/5160
MAE 4910/5910

Fast Robots
Controllability

Linear Systems Control – “review of review”

- Linear system: $\dot{x} = Ax$
- Solution: $x(t) = e^{At}x(0)$
- Eigenvectors: $T = [\xi_1 \quad \xi_2 \quad \dots \quad \xi_n]$
- Eigenvalues: $\mathrm{D} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \cdots & \\ 0 & & & \lambda_n \end{bmatrix}$
  `>>[T,D] = eig(A)`
- Linear transform: $AT = TD$
- Solution: $e^{At} = Te^{Dt}T^{-1}$
- Mapping from x to z to x: $x(t) = Te^{Dt}T^{-1}x(0)$
- Stability in continuous time: $\lambda = a + ib$, stable iff a<0
  - Discrete time: $x(k+1) = \tilde{A}x(k), \tilde{A} = e^{A\Delta t}$
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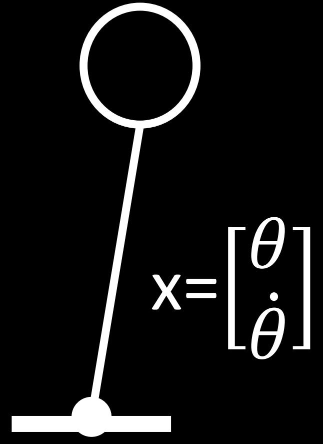
- Linearizing non-linear systems
  - Fixed points
  - Jacobian

Based on “Control Bootcamp”, Steve Brunton, UW
https://www.youtube.com/watch?v=Pi7l8mMjYVE

Linear Systems

- Linear systems review
- Eigenvectors and eigenvalues
- Stability
- Discrete time systems
- Linearizing non-linear systems
- Controllability
- LQR
- Observability

$$\dot{x} = Ax + Bu$$



$$x=\begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix}$$

This should look familiar from..
- MATH 2940 Linear Algebra
- ECE3250 Signals and systems
- ECE5210 Theory of linear systems
- MAE3260 System Dynamics
- etc…

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Controllability

Controllability

$$\dot{x} = Ax + Bu$$

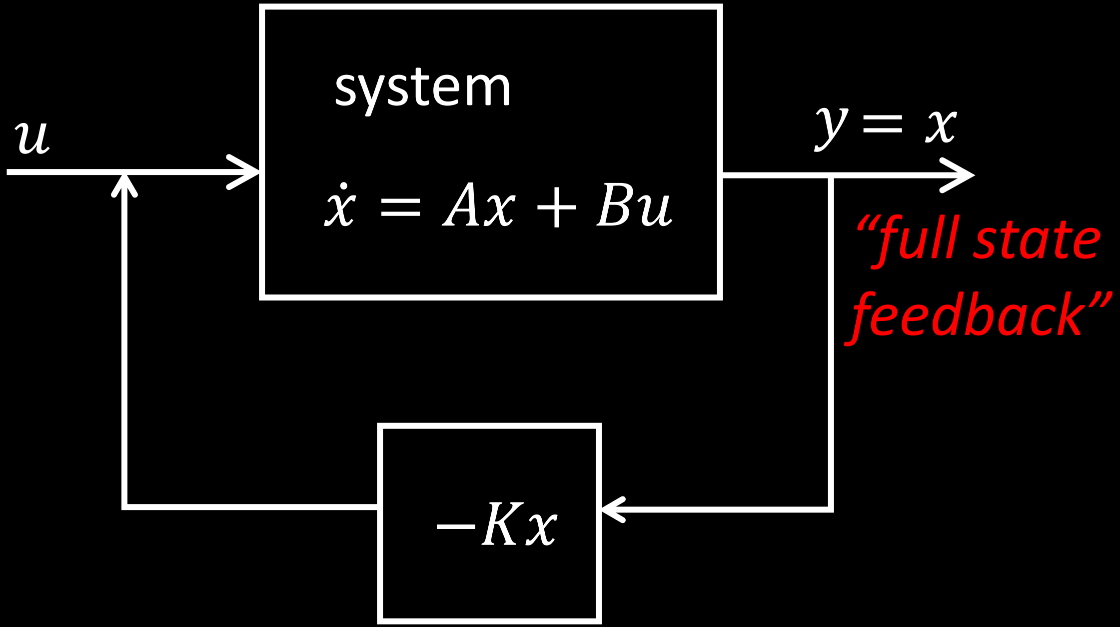
$x \epsilon \mathbb{R}^n$

$A \epsilon \mathbb{R}^{n \times m}$

$u \epsilon \mathbb{R}^q$

$B \epsilon \mathbb{R}^{n \times q}$

- Is the system controllable?
- How do we design the control law, u?



"full state feedback"

A linear controller (K matrix) can be optimal for linear systems!

Controllability

- Is the system controllable?
- How do we design the control law, u?

$$\dot{x} = Ax + Bu$$

$$\dot{x} = Ax - BKx$$

$$\dot{x} = \underbrace{(A - BK)}_{\text{New dynamics}}x$$

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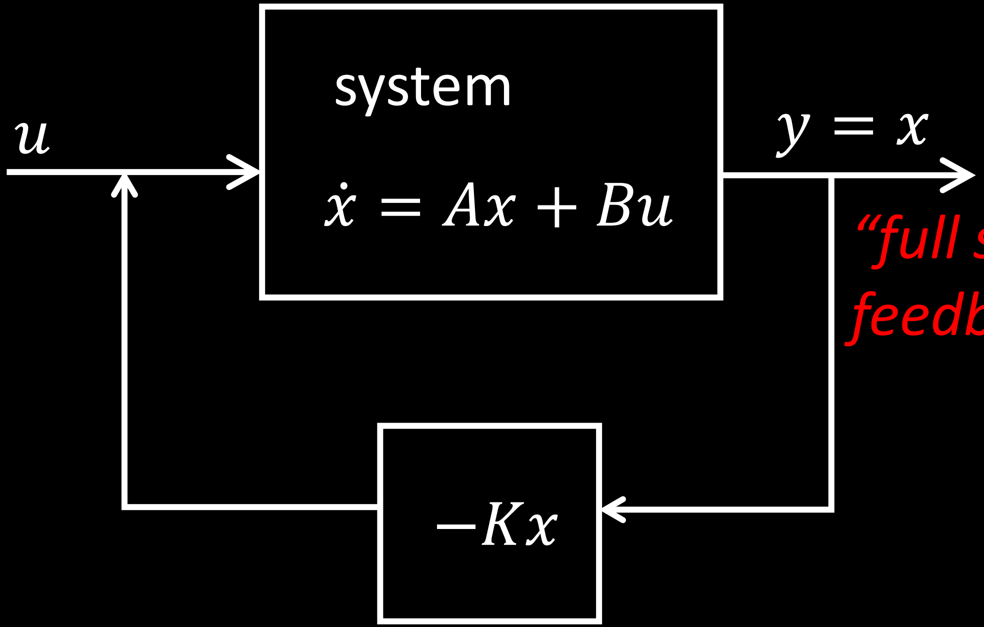

"full state feedback"

A linear controller (K matrix) can be optimal for linear systems!

Controllability

- What determines whether or not a system is controllable?
  - A system is controllable, if you can steer your state x anywhere you want in $\mathbb{R}^n$
  - Matlab >> rank(ctrb(A,B))

$$\dot{x} = Ax + Bu$$

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$$\dot{x} = \underbrace{(A - BK)}_{\text{New dynamics}} x$$

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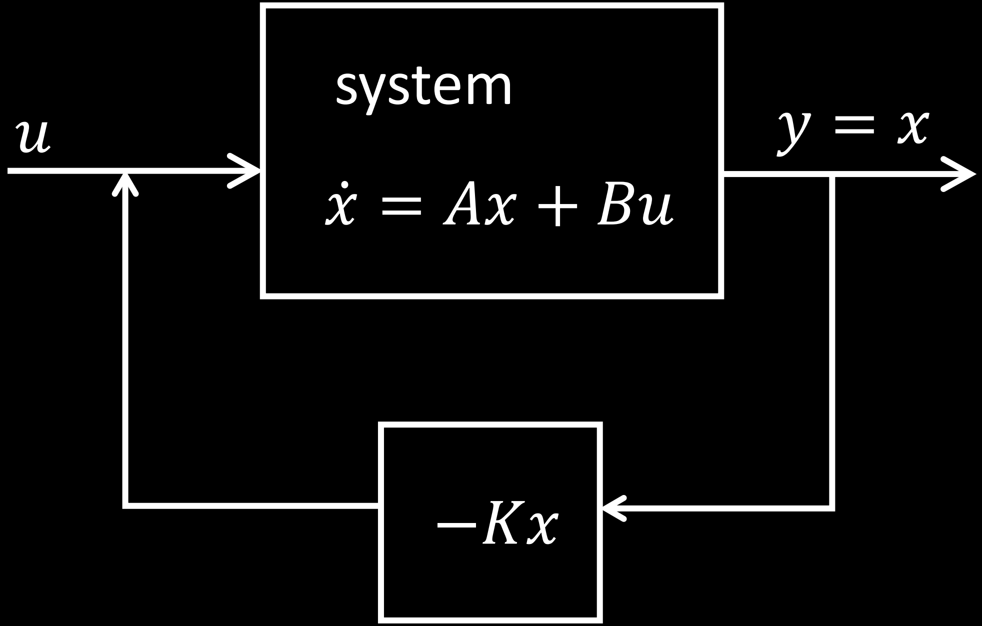

Controllability

- Can you control this system?

1. $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u$ *uncontrollable*
   - There's no way to directly/indirectly affect x_1

- What could you change to make it controllable?
  - Add more control authority!

2. $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$ *controllable*

$$\dot{x} = Ax + Bu$$

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- Can you control this system?

3. $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}1 & 1\\ 0 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u$ *controllable*
    - Systems with coupled dynamics can be controllable…
    - If A is tightly coupled, you can get away with a simple B and few sensors

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  2. $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ *controllable*
  3. $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$ *controllable*
- Controllability matrix
  - Matlab >> ctrb(A,B)
  - $\mathbb{C} = [B \quad AB \quad A^2B \quad \ldots \quad A^{n-1}B]$
  - Iff $\mathrm{rank}(\mathbb{C}) = n$ the system is controllable

$$\dot{x} = Ax + Bu$$

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  - $\mathbb{C} = [B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B]$
  - Iff $\text{rank}(\mathbb{C}) = n$ the system is controllable
- System 1:
  - $\mathbb{C} = \left[\qquad \right]$ ← *These still can't touch x_1!*
  - $\mathbb{C} = \begin{bmatrix}0 & 0\\ 1 & 2\end{bmatrix}$ rank=1, n=2

$$\dot{x} = Ax + Bu$$

$$\dot{x} = Ax - BKx$$

$$\dot{x} = \underbrace{(A - BK)}_{\text{New dynamics}}x$$

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- Controllability matrix
  - Matlab >> ctrb(A,B)
  - $\mathbb{C} = [B \quad AB \quad A^2B \quad \ldots \quad A^{n-1}B]$
  - Iff $\text{rank}(\mathbb{C}) = n$ the system is controllable
- System 1: $\mathbb{C} = \begin{bmatrix}0 & 0\\ 1 & 2\end{bmatrix}$ rank=1, n=2
- System 3:
  - $\mathbb{C} = \begin{bmatrix}0 & 1\cdot 0 + 1\cdot 1\\ 1 & 0\cdot 0 + 2\cdot 1\end{bmatrix} = \begin{bmatrix}0 & 1\\ 1 & 2\end{bmatrix}$ rank=2, n=2

$$\dot{x} = Ax + Bu$$

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$$\dot{x} = \underbrace{(A - BK)}_{\text{New dynamics}}x$$

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Fyi!

- Just because a linearized, nonlinear system is uncontrollable, it can still be nonlinearly controllable!

Controllability Matrix and the Discrete Time Impulse Response

$$\dot{x} = Ax + Bu, \qquad x \in \mathbb{R}^n$$

$$\mathbb{C} = [B \quad AB \quad A^2B \quad \ldots \quad A^{n-1}B]$$

- Why does $\mathbb{C}$ predict controllability?!
- Discrete time impulse response: $x(k+1) = \tilde{A}x(k) + \tilde{B}u(k)$

(assume a single input actuator)

  - $u(0) = 1 \qquad x(0) = 0$
  - $u(1) = 0 \qquad x(1) = \tilde{A}x(0) + \tilde{B}u(0) = \tilde{B}$
  - $u(2) = 0 \qquad x(2) = \tilde{A}x(1) + \tilde{B}u(0) = \tilde{A}\tilde{B}$
  - $u(3) = 0 \qquad x(3) = \tilde{A}^2\tilde{B}$
  - $\ldots \qquad \ldots$
  - $u(m) = 0 \qquad x(m) = \tilde{A}^{m-1}\tilde{B}$

If the system is controllable, then the impulse response affects every state in $\mathbb{R}^n$

Linear Systems Control – "review of review"

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- Linearizing non-linear systems
  - Fixed points
  - Jacobian
- Controllability
  - $\dot{x} = (A - BK)x$
  - `>>rank(ctrb(A,B))`

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Reachability

Controllabillity and Reachability

$$\dot{x} = Ax + Bu, \qquad x \in \mathbb{R}^n$$

$$\mathbb{C} = [B \quad AB \quad A^2B \quad \ldots \quad A^{n-1}B]$$

Reachability

- $\mathcal{R}_t$, states that are reachable at time t
- $\mathcal{R}_t = \{\xi \epsilon \mathbb{R}^n$ for which there is an input $u(t)$ that makes $\mathrm{x(t)} = \xi$

Equivalences

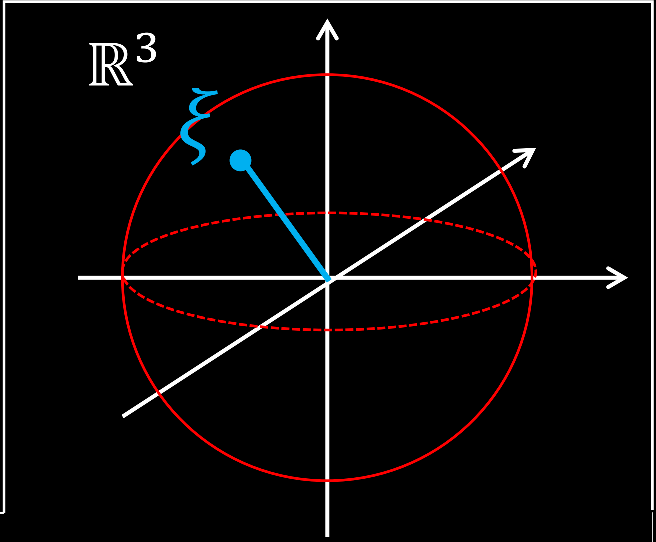
1. The system is controllable
    - iff $rank(\mathbb{C}) = n$
2. You can choose K to arbitrarily place the eigenvalues of your closed loop system
    - $\dot{x} = (A - BK)x$
3. You can reach anywhere in $\mathbb{R}^n$ in a finite amount of time and energy
    - $\mathcal{R}_t = \mathbb{R}^n$

Controllabillity and Reachability

$$\dot{x} = Ax + Bu, \qquad x \in \mathbb{R}^n$$

$$\mathbb{C} = [B \quad AB \quad A^2B \quad \ldots \quad A^{n-1}B]$$

(if the point is reachable, any point in that direction is reachable)



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    - $\dot{x} = (A - BK)x$ *
3. You can reach anywhere in $\mathbb{R}^n$ in a finite amount of time and energy
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>>K = scipy.signal.place_poles(*A*, *B*, *poles*)

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Controllability Gramians

- We can test if the system is controllable
- But not how easy it is to control
- …or which directions are the easiest
- …or how we could best improve our control authority

Controllabillity Gramian

- $x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} Bu(\tau)d\tau$

- Controllability Gramian
  - $W_t = \int_0^t e^{A\tau}BB^T e^{A^T\tau}d\tau \quad W_t \epsilon \mathbb{R}^{nxn}$
  - Discrete time
    - $W_t \approx \mathbb{C}\mathbb{C}^T$
    - $W_t \xi = \lambda \xi$

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n$$

$$\mathbb{C} = [B \quad AB \quad A^2B \quad \ldots \quad A^{n-1}B]$$

```
>> rank(ctrb(A,b))
>> [U,S,V] = svd(ℂ, ‘econ’)
```

The SVD of A takes the form: $A = U\Sigma V^T$
U = left singular vector
V = right singular vector
Σ = diagonal matrix with singular values

(The eigenvectors with the biggest eigenvalues of the controllability gramian, are also the most controllable directions in state space!)

Controllabillity Gramian

- $x(t) = e^{At}x(0) \; + \int_0^t e^{A(t-\tau)}\, Bu(\tau)d\tau$

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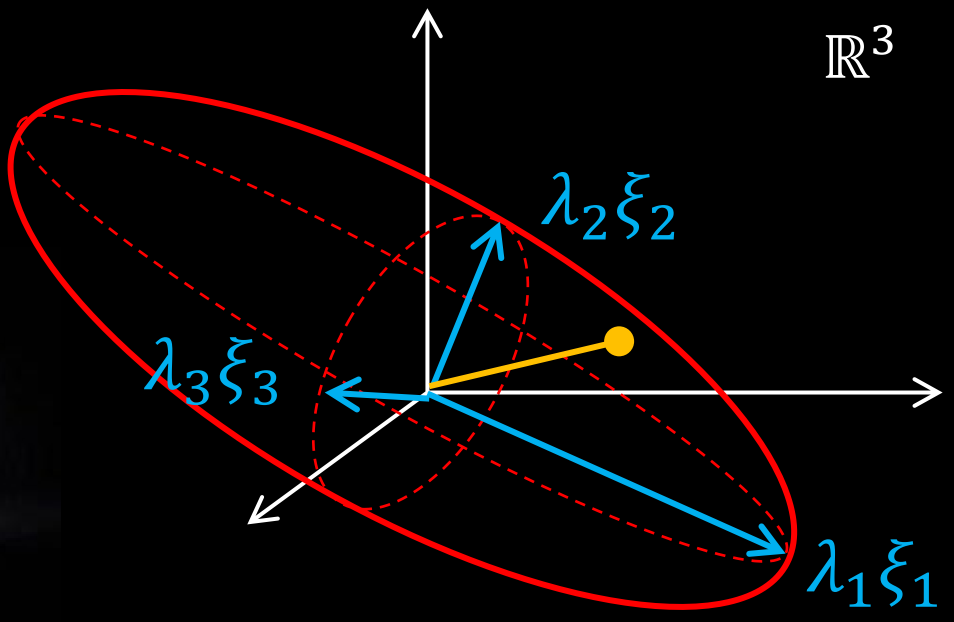

Controllabillity Gramian

By DLR, CC-BY 3.0, CC BY 3.0 de, https://commons.wikimedia.org/w/index.php?curid=61072555

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- Controllability for very high dimensional systems?
- Many directions in $\mathbb{R}^n$ are extremely stable - you only need to control directions that impact your control objective
- *Stabilizability*

Controllabillity Gramian

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- $x(t) = e^{At}x(0) \ + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$

  (convolution of e^{At} with $u(\tau)$)

- Controllability Gramian
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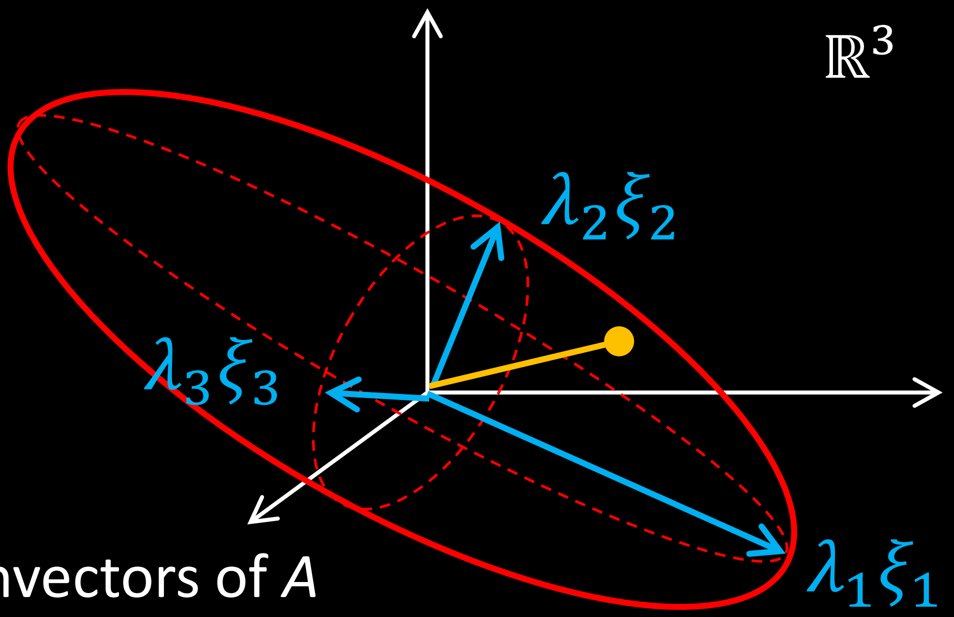

…and lightly damped

- *Stabilizability*
  - A system is stabilizable iff all unstable eigenvectors of *A* are in the controllable subspace

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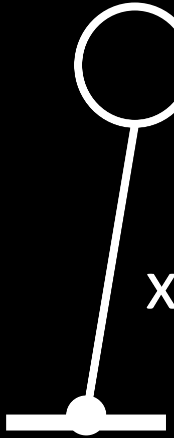
Linear Systems

- Linear systems review
- Eigenvectors and eigenvalues
- Stability
- Discrete time systems
- Linearizing non-linear systems
- Controllability
- LQR control
- Observability

$$\dot{x} = Ax + Bu$$



$$x=\begin{bmatrix}\theta \\ \dot{\theta}\end{bmatrix}$$

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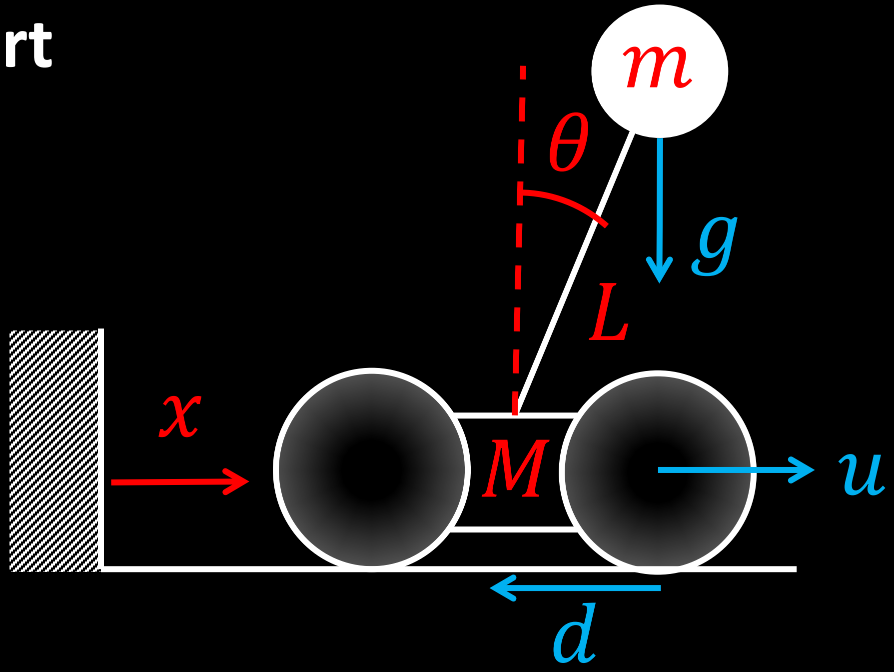
Inverted Pendulum on a Cart

Based on Steve Brunton’s Control Bootcamp lecture series

Inverted Pendulum on a Cart



Force acting on the cart in the x direction

Eq. of motion

↓

State space model → Fixed points, $\bar{x}$ → Jacobian → (A,B) Controllable?

$$y = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix}$$

down

$\theta = 0, \pi$

up

$\dot{\theta} = 0$

$\dot{x} = 0$

x free variable

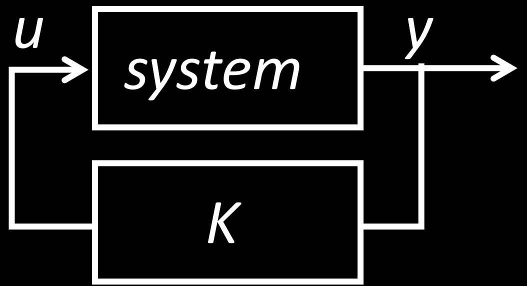
$$\frac{df}{dy}\Big|_{\bar{y}}$$

$$\dot{y} = Ay + Bu$$

↓

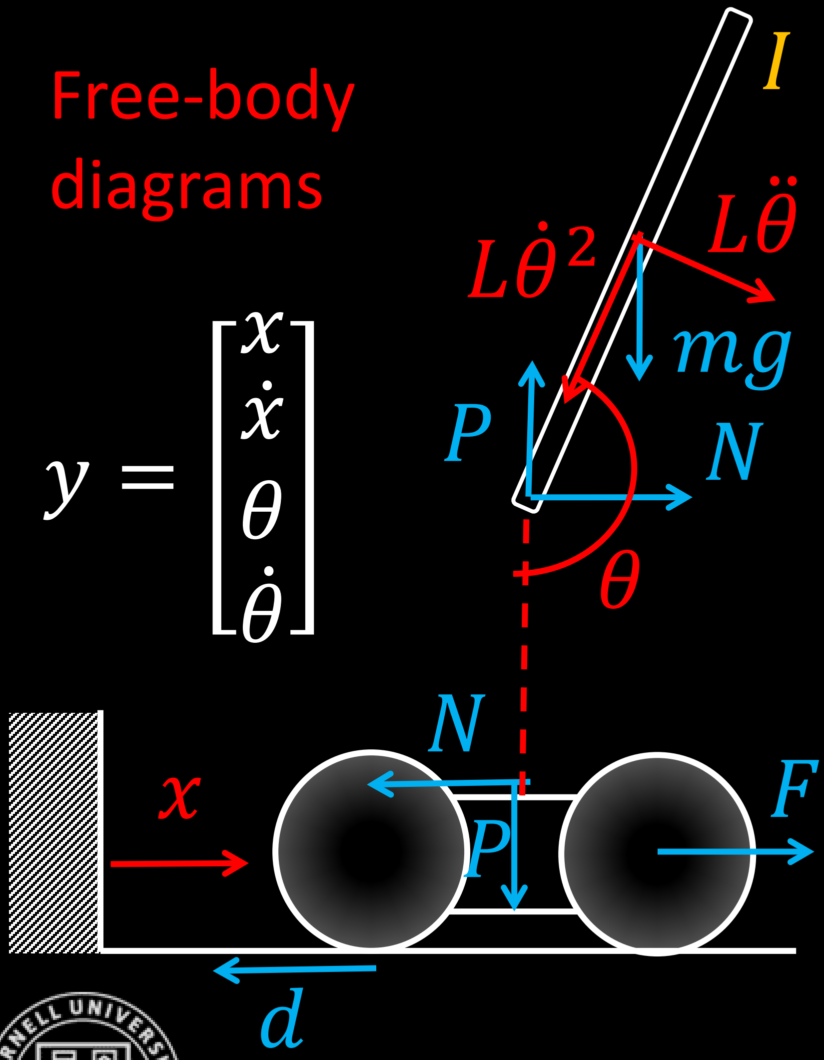
Add linear control

$$\dot{y} = (A - BK)y$$



Inverted Pendulum on a Cart – State Space equations

Free-body diagrams



$$y = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix}$$

$$\ddot{\varphi} = \frac{(M+m)g}{ML}\varphi - \frac{d}{ML}\dot{x} + \frac{1}{ML}u \qquad \ddot{x} = \frac{m}{M}g\varphi - \frac{d}{M}\dot{x} + \frac{1}{M}u$$

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-(I+ml^2)b}{I(M+m)+Mml^2} & \frac{m^2gl^2}{I(M+m)+Mml^2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-mlb}{I(M+m)+Mml^2} & \frac{mgl(M+m)}{I(M+m)+Mml^2} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \phi \\ \dot{\phi} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{(I+ml^2)b}{I(M+m)+Mml^2} \\ 0 \\ \frac{-ml}{I(M+m)+Mml^2} \end{bmatrix} \mathrm{u}$$

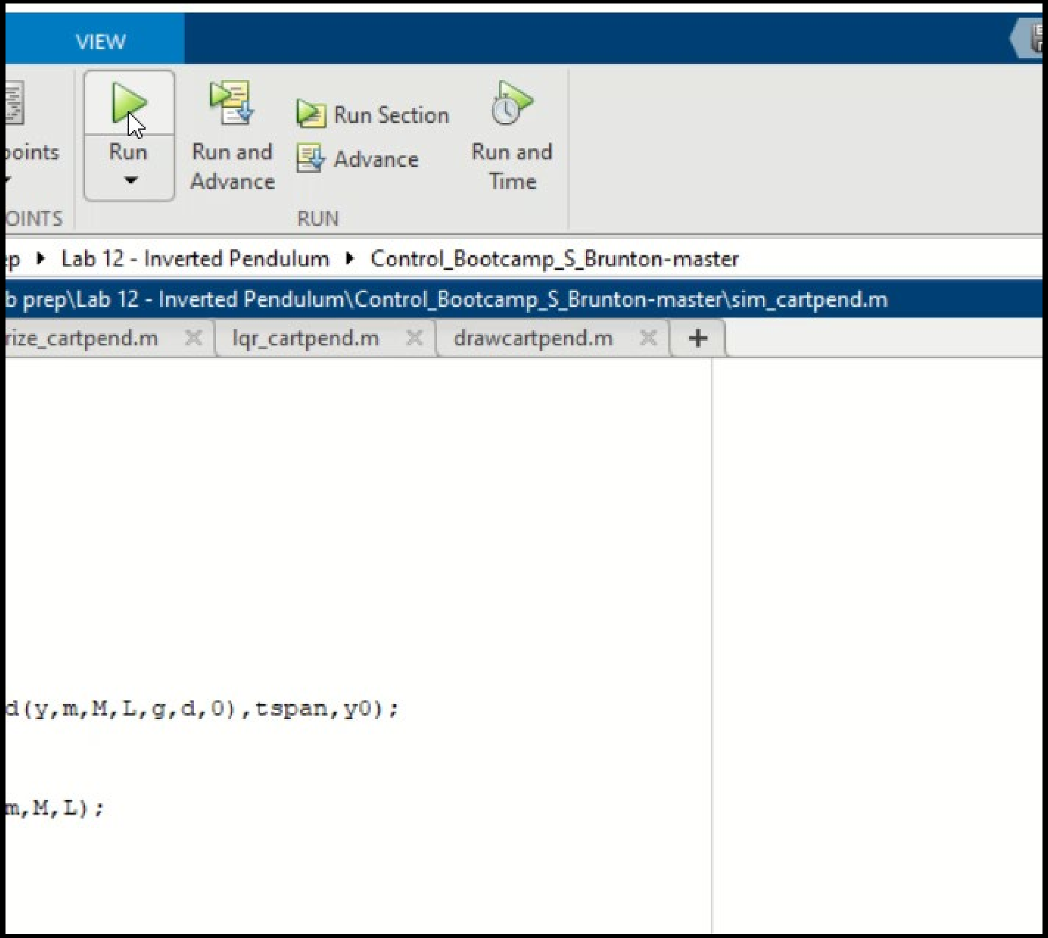
Linearized about fixed point ($\theta = \pi$)

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{d}{M} & \frac{m}{M}g & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{d}{ML} & \frac{(M+m)g}{ML} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{ML} \end{bmatrix} u$$

Inverted Pendulum on a Cart

Matlab example

- Non-linear model
- Linearized model
- Eigenvalues
- Stability
- Controllability



b prep\Lab 12 - Inverted Pendulum\Control_Bootcamp_S_Brunton-master\sim_cartpend.m

```
d(y,m,M,L,g,d,0),tspan,y0);


m,M,L);
```

$\gg eig(A)$

$\lambda_4 = 3.5069$

$\lambda_3 = -1.9278$

$\lambda_2 = -3.6844$

$\lambda_1 = 0$

$\gg rank(ctrb(A,B))$

4

Inverted Pendulum on a Cart

Matlab example

- Non-linear model
- Linearized model
- Eigenvalues
- Stability
- Controllability
- Add control

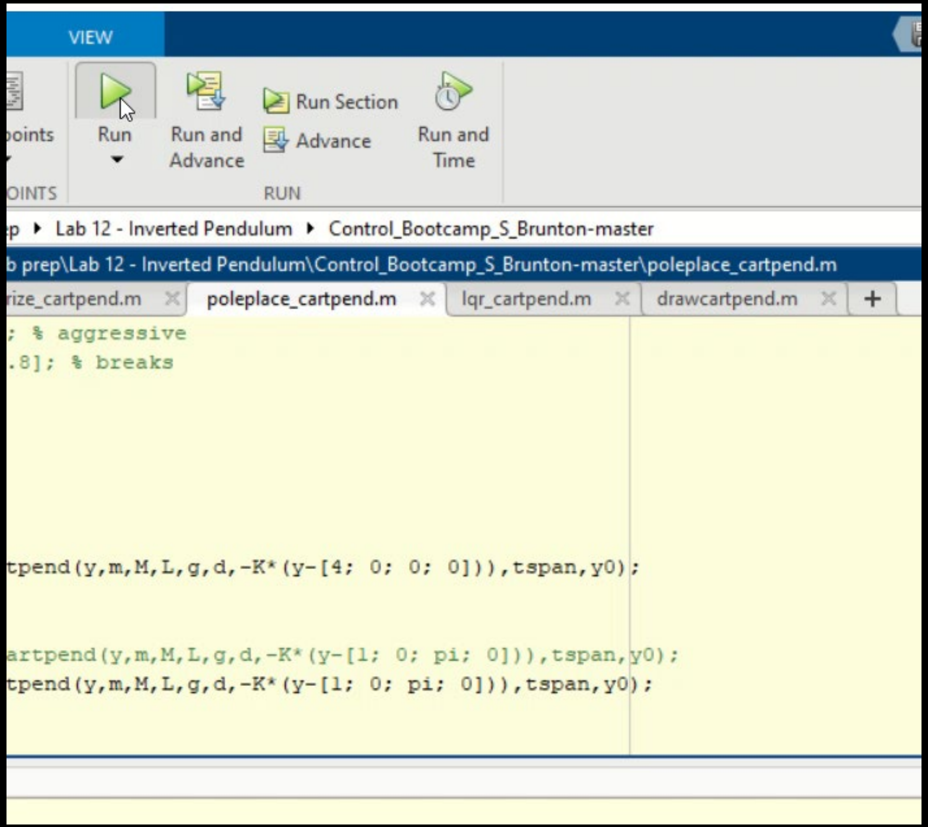
$\gg eigs = [-1.1; -1.2; -1.3; -1.4]$

$\gg K = place(A,B,eigs)$

$K=[-0.0965 \quad -1.3111 \quad 8.7254 \quad 2.2295]$

$\gg eig(A\text{-}B.{*}K)$

$[-1.4; -1.3; -1.2; -1.1]$



```
; % aggressive
.8]; % breaks







tpend(y,m,M,L,g,d,-K*(y-[4; 0; 0; 0])),tspan,y0);


artpend(y,m,M,L,g,d,-K*(y-[1; 0; pi; 0])),tspan,y0);
tpend(y,m,M,L,g,d,-K*(y-[1; 0; pi; 0])),tspan,y0);
```

Pole Placement

- In Python
  - https://docs.scipy.org/doc/scipy/reference/generated/scipy.signal.place_poles.html
  - K = scipy.signal.place_poles(A, B, poles)
- Barely stable eigenvalues
  - Not enough control authority
- More negative eigenvalues
  - Faster dynamics
  - Less robust system
- Linear Quadratic Control (LQR)
  - “Sweet spot of eigenvalues”
  - Balances how fast you stabilize your state and how much control energy you spend to get there

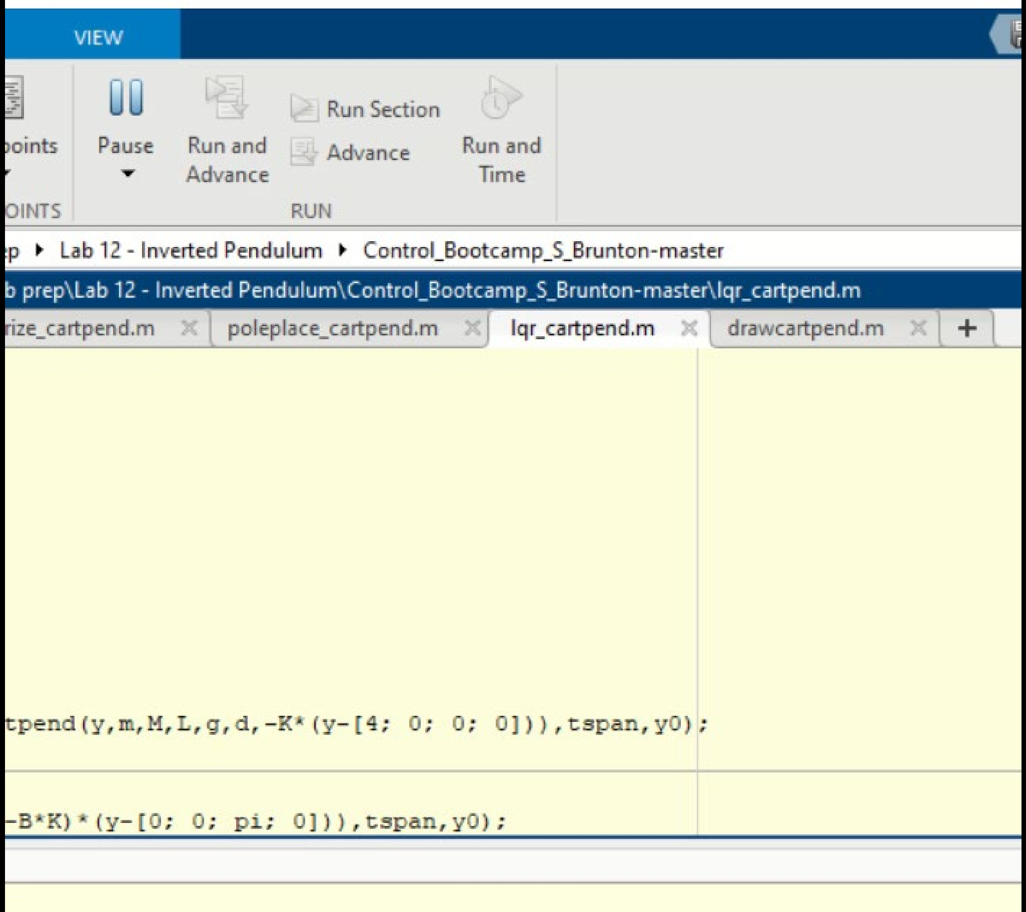
Linear Quadratic Control

$$\dot{x} = Ax + Bu, x \epsilon \mathbb{R}^n$$
$$u = -Kx$$
$$\dot{x} = (A - BK)x$$

- >> K = place(A,B,eigs)
- Where are the best eigs??
  - Linear Quadratic Regulator (LQR)
    - >> K = lqr(A,B,Q,R)
    - $Q = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 10 & \\ 0 & & & 100 \end{bmatrix}, R = 0.001$
    - Ricotta equation
      - $\int_0^\infty (x^T Q x + u^T R u) dt$
      - Computationally expensive, O(n^3)

Matlab Example

- $Q = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 10 & \\ 0 & & & 100 \end{bmatrix}, R = 0.001$
- >>K = lqr(A,B,Q,R);
- >>[T,D] = eigs(A-B.*K)
  - λ_1= -788.29 + 0.00i
  - λ_2= -0.70 + 0.83i
  - λ_3=-0.70 - 0.83i
  - λ_4=-0.83 + 0.00i
- >>T(:,1)
  - = [0.0008, -0.6387, 0.0010, -0.7695]^T



```
tpend(y,m,M,L,g,d,-K*(y-[4; 0; 0; 0])),tspan,y0);

-B*K)*(y-[0; 0; pi; 0])),tspan,y0);
```

Matlab Example

- $Q = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 10 & \\ 0 & & & 100 \end{bmatrix}, R = 0.001$
- >>K = lqr(A,B,Q,R);
- >>[T,D] = eigs(A-B.*K)
  - λ_1= -788.29 + 0.00i
  - λ_2= -0.70 + 0.83i
  - λ_3=-0.70 - 0.83i
  - λ_4=-0.83 + 0.00i
- >>T(:,1)
  - = $[0.0008, -0.6387, 0.0010, -0.7695]^T$

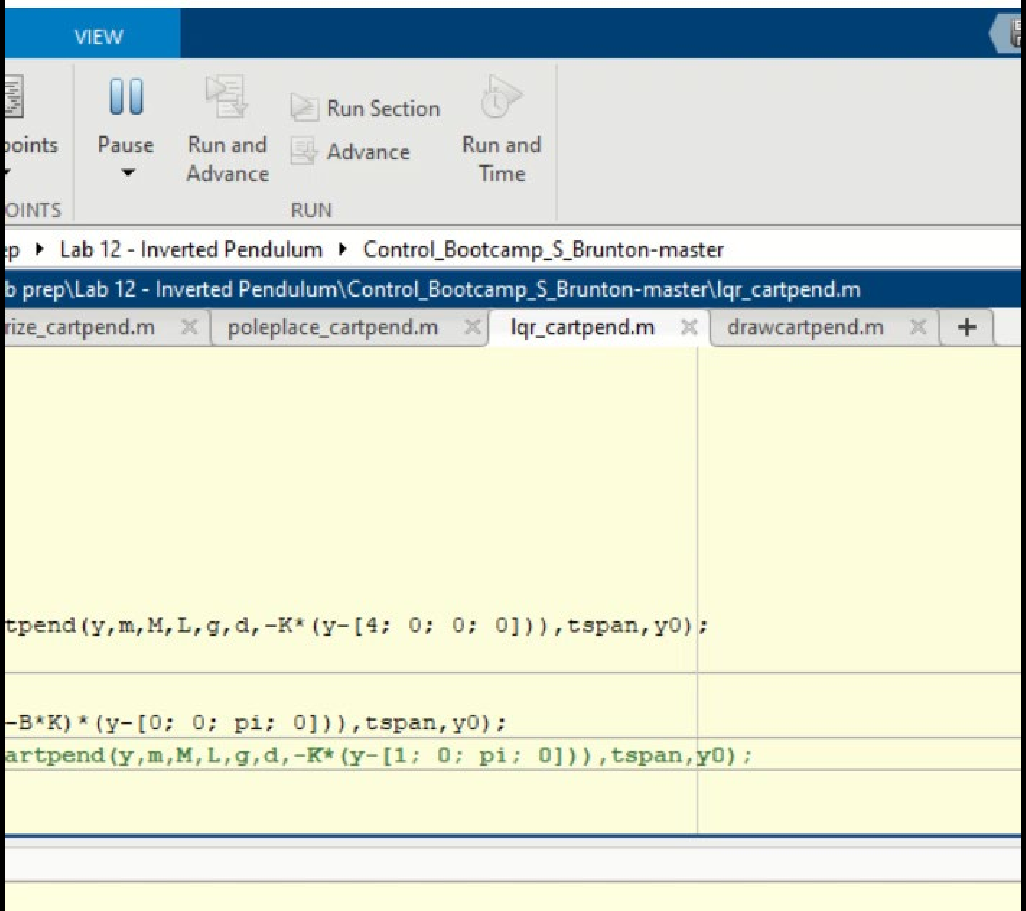
$R = 1$

λ_1 = -25.6851 + 0.0000i

λ_2 = -1.0855 + 0.8921i

λ_3 = -1.0855 - 0.8921i

λ_4 = -0.4811 + 0.0000i



```
tpend(y,m,M,L,g,d,-K*(y-[4; 0; 0; 0])),tspan,y0);

-B*K)*(y-[0; 0; pi; 0])),tspan,y0);
artpend(y,m,M,L,g,d,-K*(y-[1; 0; pi; 0])),tspan,y0);
```

Linear Quadratic Control

- >> K = place(A,B,eigs)
- Where are the best eigs??
  - Linear Quadratic Regulator (LQR)
    - >> K = lqr(A,B,Q,R)
    - $\int_0^\infty (x^T Q x + u^T R u) dt$
    - $Q = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 10 & \\ 0 & & & 100 \end{bmatrix}, R = 0.001$
    - Riccati equation
      - Computationally expensive, O(n^3)

$$\dot{x} = Ax + Bu, x \epsilon \mathbb{R}^n$$
$$u = -Kx$$
$$\dot{x} = (A - BK)x$$

- *The linear controller works!*
  - *(in simulation)*
- *Issues in Practice?*
  - *Imperfect models*
  - *Nonlinear parts*
    - *Deadband, saturation, etc.*
  - *Partial state feedback*

Linear Systems Control – “review of review”

- Linear system: $\dot{x} = Ax$
- Solution: $x(t) = e^{At}x(0)$
- Eigenvectors: $T = [\xi_1 \quad \xi_2 \quad \dots \quad \xi_n]$
- Eigenvalues: $$\mathrm{D} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \dots & \\ 0 & & & \lambda_n \end{bmatrix}$$
  `>>[T,D] = eig(A)`
- Linear transform: $AT = TD$
- Solution: $e^{At} = Te^{Dt}T^{-1}$
- Mapping from x to z to x: $x(t) = Te^{Dt}T^{-1}x(0)$
- Stability in continuous time: $\lambda = a + ib$, stable iff a<0
  - Discrete time: $x(k+1) = \tilde{A}x(k), \tilde{A} = e^{A\Delta t}$
    - Stability in discrete time: $\tilde{\lambda}^n = R^n e^{in\theta}$, stable iff R<1

- Linearizing non-linear systems
  - Fixed points
  - Jacobian
- Controllability
  - $\dot{x} = (A - BK)x$
  - `>>rank(ctrb(A,B))`
- Reachability
- Controllability Gramian
- Pole placement
  - `>>K=place(A,B,p)`
- Optimal control (LQR)
  - `>>K=lqr(A,B,Q,R)`